History-Based Problems in Senior High School Mathematics Teaching in Mainland China

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- History-Based Problem Posing
- History-Based Problems in 20 HPM Lessons
- Distribution of History-Based Problems in Different Parts of the HPM Lessons
- Functions of History-Based Problems in HPM Lessons
- Conclusions

Backgrounds

- In mainland China, more and more high school mathematics teachers are interested in HPM, however, they seldom actually use history in teaching.
- Generally speaking, high school students, who are burdened with the huge pressure of national college entrance examination, have little opportunity to learn about the history of mathematics.

Backgrounds

A Classroom in Beijing, June 8, 2010, after the college entrance examination

Backgrounds

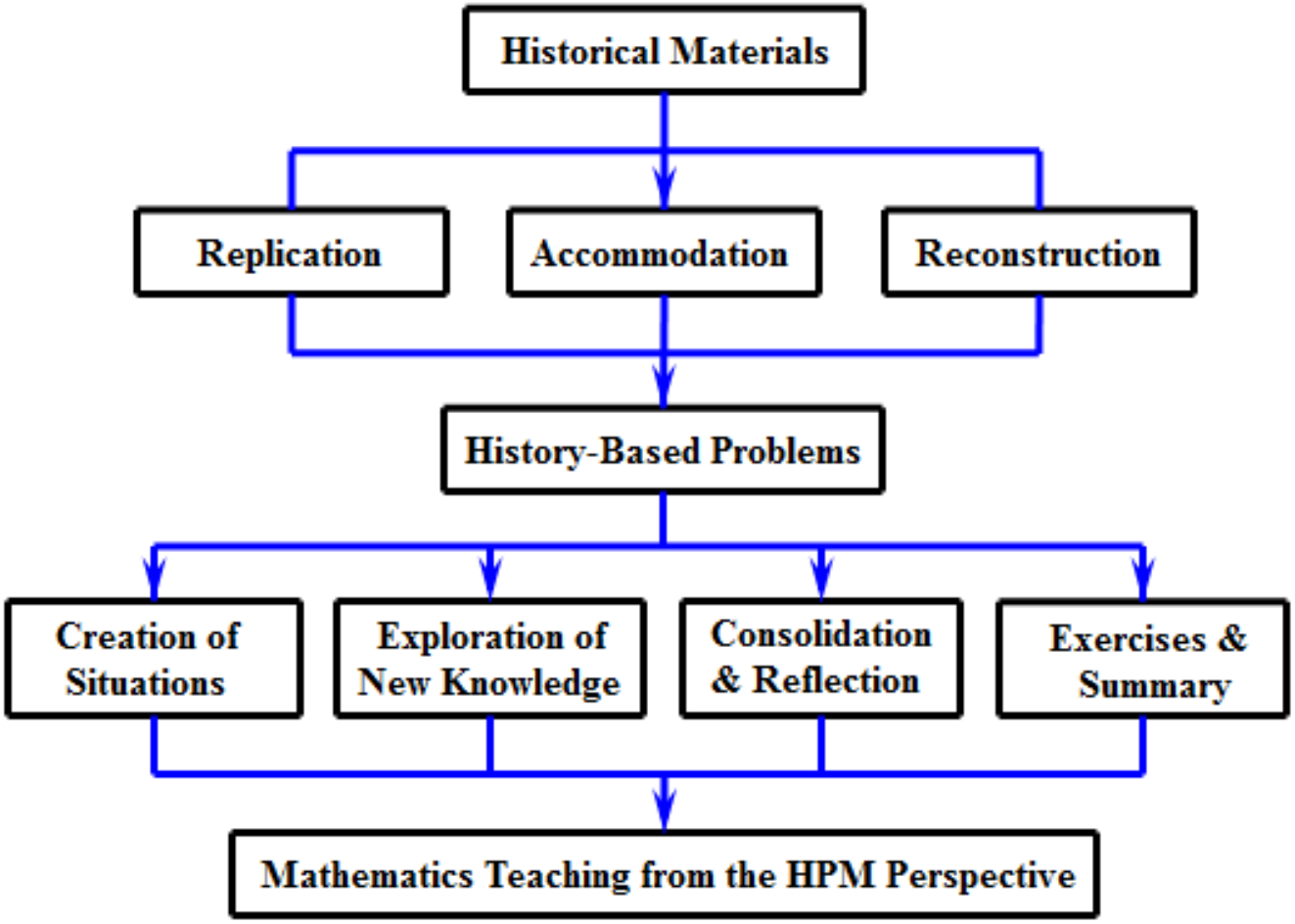
- In mainland China, researches in HPM have focused on teaching practice and mathematics teachers' professional development and **many lessons in which the history of mathematics is integrated** (abbreviated as **HPM lessons**) have been developed in recent years.
- Four approaches to using history in mathematics teaching are adopted in these lessons.
- Most of historical materials used are mathematical problems and problem posing is involved in all approaches except the complementation one.

Backgrounds

Approaches	Teaching Materials
Complementation	Pictures, stories, etc.
Replication	Original problems, methods, etc. in the history, copied without any adaption
Accommodation	Problems adapted from original ones or based on historical information
Reconstruction	The evolution of a subject (e.g. a concept or theory) inspired by the history of mathematics

In this talk, mathematical problems posed on the basis of historical information are called history-based problems.

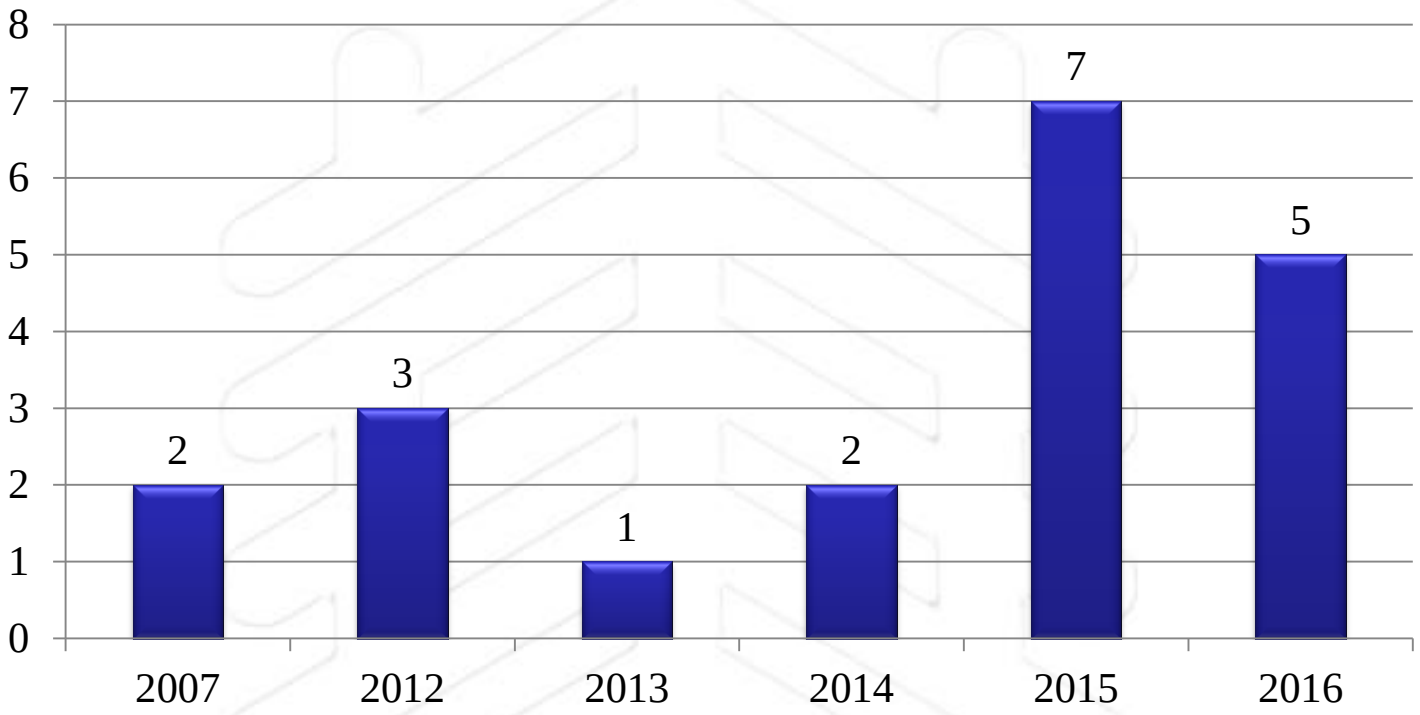
Formation of History-Based Problems and their Uses in HPM Lessons



Research Questions

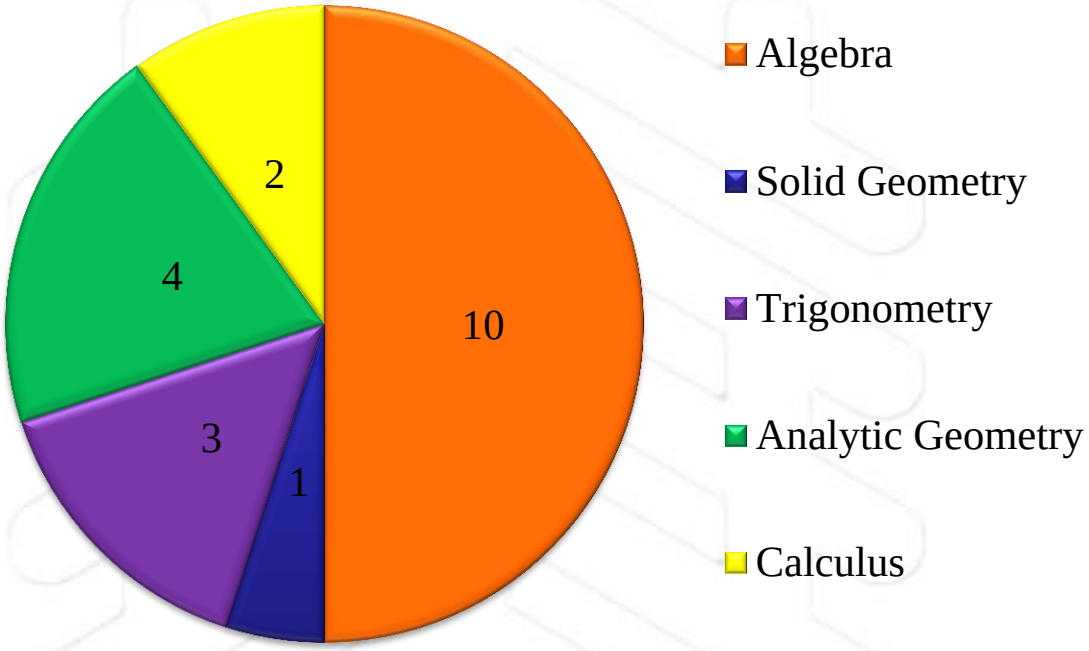
- What kind of history-based problems are used in HPM lessons already developed in Mainland China?
- How are they posed?
- How are they distributed in different parts of the HPM lessons?
- What roles do they play in these lessons?

Selection of HPM Lessons



20 HPM Lessons Developed from 2007 to 2016

Selection of HPM Lessons

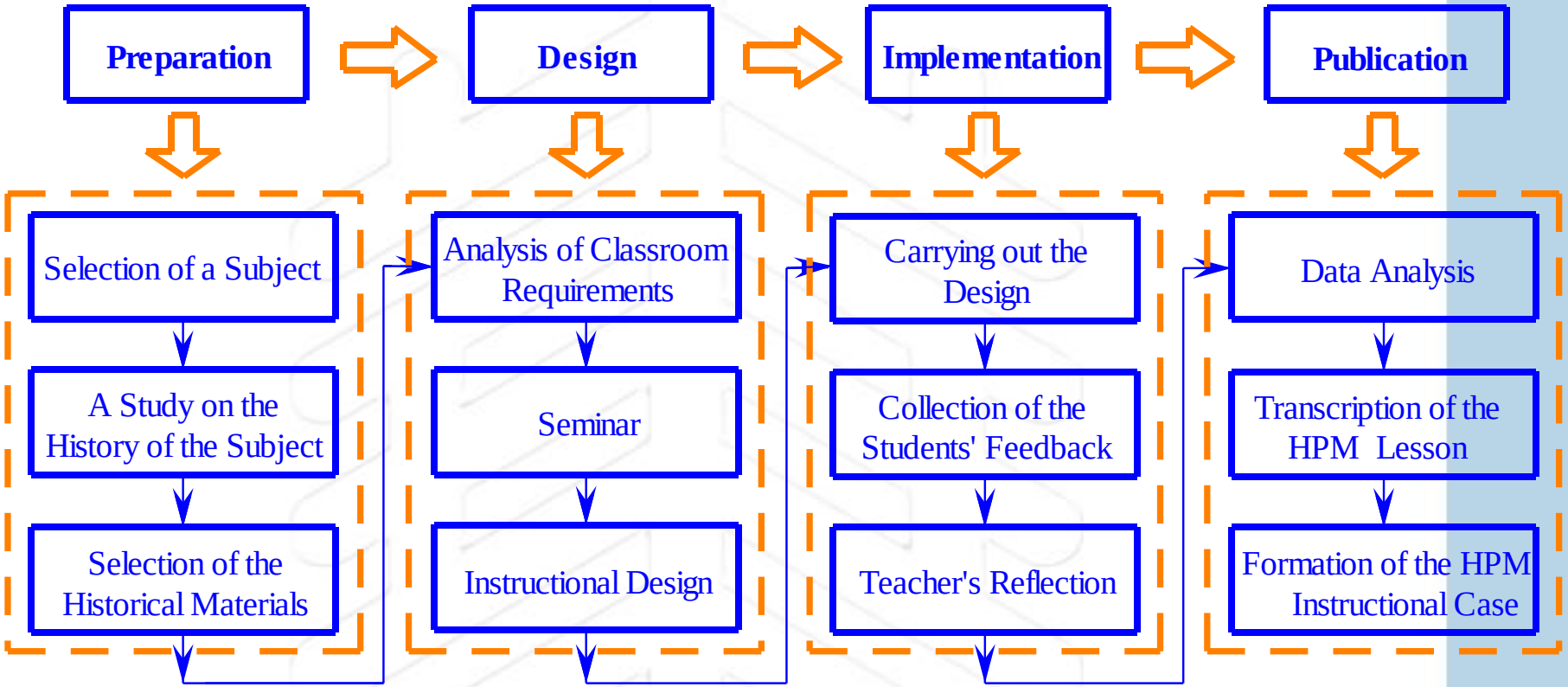


Subject Distribution of 20 HPM Lessons

Selection of HPM Lessons

Areas	The HPM Lessons	Grade
Algebra	The concept of function;	10
	The concept of logarithm (I-II);	10
	The mean value inequality;	10
	Zeros of a function and solutions of an equation;	10
	The concept of number sequence;	11
	The recursive number sequence;	11
	The sum of a geometrical sequence;	11
	Introduction of complex numbers (I-II)	12
Solid Geometry	The definition of a prism	11
Analytic Geometry	Curves and their equations;	11
	The concept of slope;	11
	The definition and equation of an ellipse;	11
	The definition and equation of a parabola	11
Trigonometry	The addition & subtraction formulas for sine and cosine;	11
	The law of cosine;	11
	The law of sine	11
Calculus	The geometrical meaning of derivatives;	12
	The application of derivatives	12

Selection of HPM Lessons

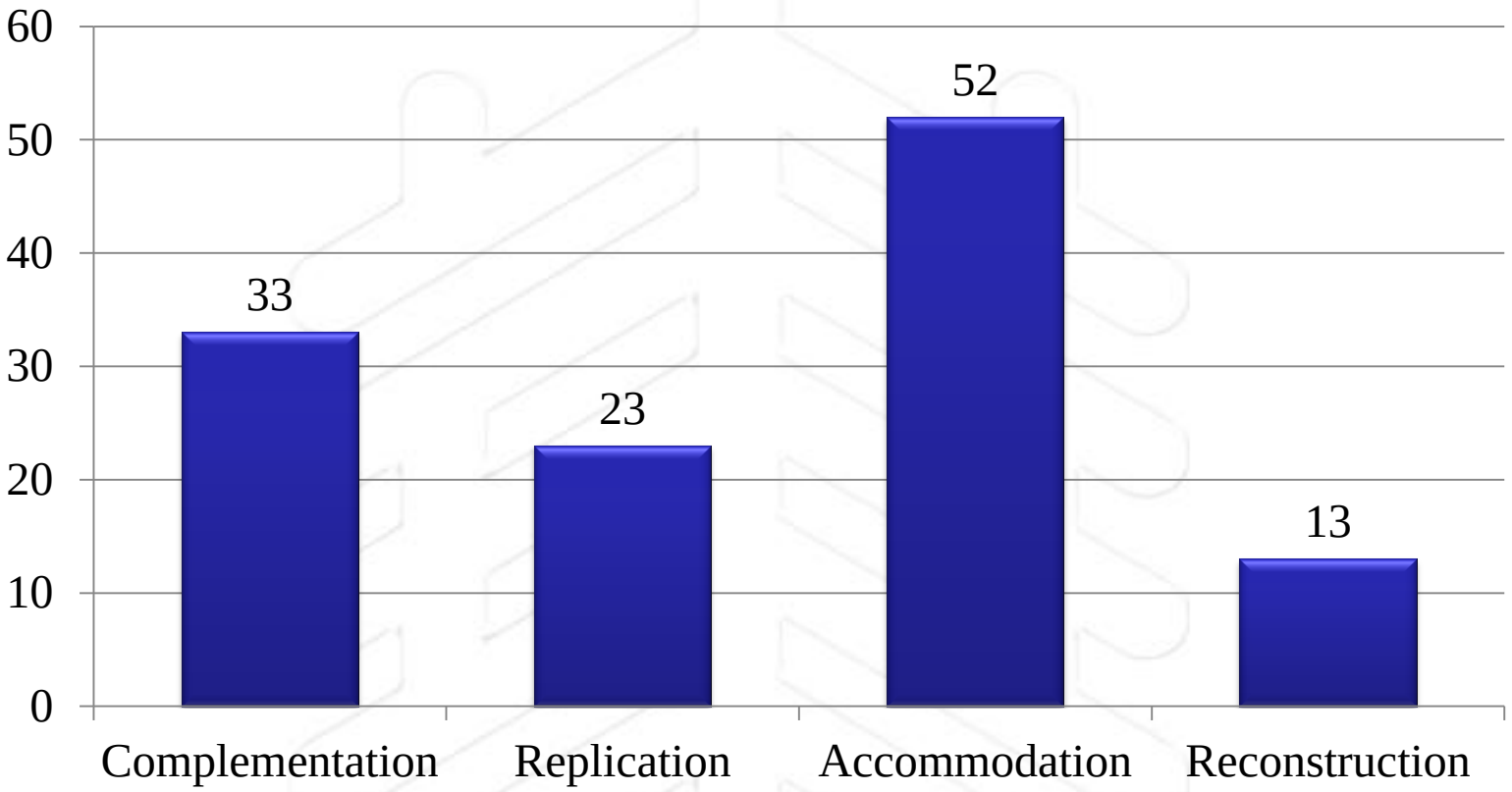


The Process of Development of HPM Lessons

Selection of HPM Lessons

Seminars on Instructional Designs of HPM Lessons

Selection of HPM Lessons



The Frequency Distribution Histogram of the Four Approaches Adopted in 20 HPM Lessons

History-Based Problem Posing

Silver et al (1996) identified **four strategies** of posing new problems in terms of the given situations or problems:

- **constraint manipulation** (i.e., systematic manipulation of the task conditions or implicit assumptions);
- **goal manipulation** (i.e., manipulation of the goal of a given or previously posed problem where the assumptions of the problem are accepted with no change);
- **symmetry** (i.e., a symmetrical exchange between the existing problem's goal and conditions);
- **chaining** (i.e., expanding an existing problem so that a solution to the new problem would require solving the existing one first).

Seven strategies of **history-based problem posing (HBPP)** are identified:

- **Copying**, a strategy of **HBPP** by means of **directly using original problems without any adaption**.

*A **c**opied **o**riginal **problem** is abbreviated as **a CO Problem**.*

- **Free posing**, a strategy of **HBPP** which, starting from some historical information (stories, problems, methods, etc.), selects conditions and goals according to classroom requirements.

*A problem posed with **f**ree **p**osing strategy is abbreviated as **a FP Problem**.*

Seven strategies of **history-based problem posing (HBPP)** are identified:

- **Situation manipulation**, a strategy of **HBPP** by means of **adapting the situation of or adding a realistic situation to the historical problem with the given conditions and goals remaining unchanged**.

*A problem posed with **s**ituation **m**anipulation strategy is abbreviated as **a SM Problem**.*

Seven strategies of **history-based problem posing (HBPP)** are identified:

- **Constraint manipulation**, a strategy of **HBPP** by means of **changing the constraints/conditions of a historical problem with the goals remaining unchanged**.

*A problem posed with **c**onstraint **m**anipulation strategy is abbreviated as **a CM Problem**.*

Seven strategies of **history-based problem posing (HBPP)** are identified:

- **Goal manipulation**, a strategy of **HBPP** by means of **changing the goals of a historical problem with the constraints/conditions remaining unchanged**.

*A problem posed with **g**oal **m**anipulation strategy is abbreviated as **a GM Problem**.*

Seven strategies of **history-based problem posing (HBPP)** are identified:

- **Symmetry**, a strategy of **HBPP** by means of **symmetrically exchanging conditions and goals of a historical problem**.

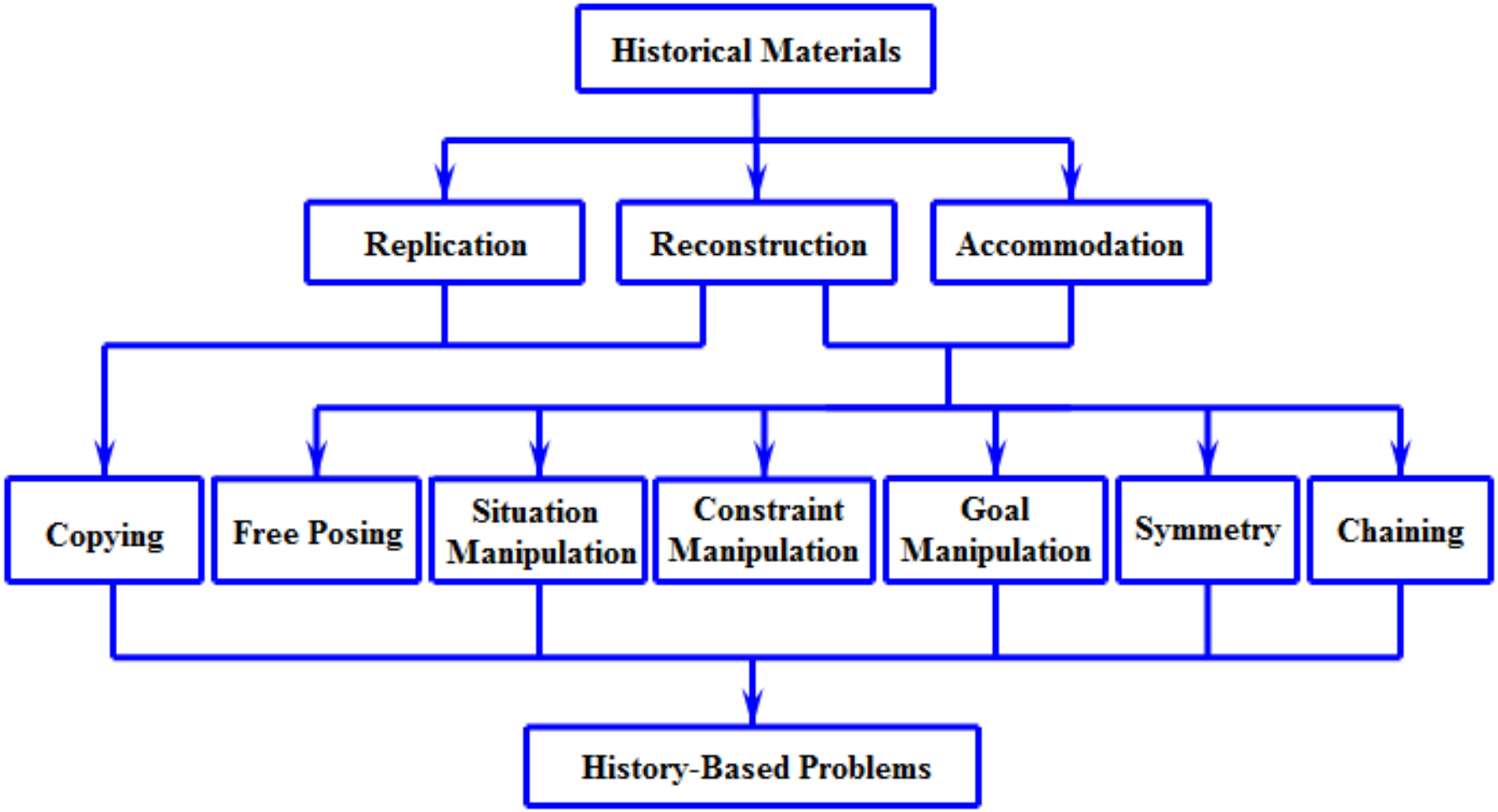
*A problem posed with **s**ymmetrical **e**xchange strategy is abbreviated as **a SE Problem**.*

Seven strategies of **history-based problem posing (HBPP)** are identified:

- **Chaining**, a strategy of **HBPP** by means of **taking the goals of the original problems as the conditions**.

*A problem posed with **ch**aining strategy is abbreviated as **a CH Problem***.

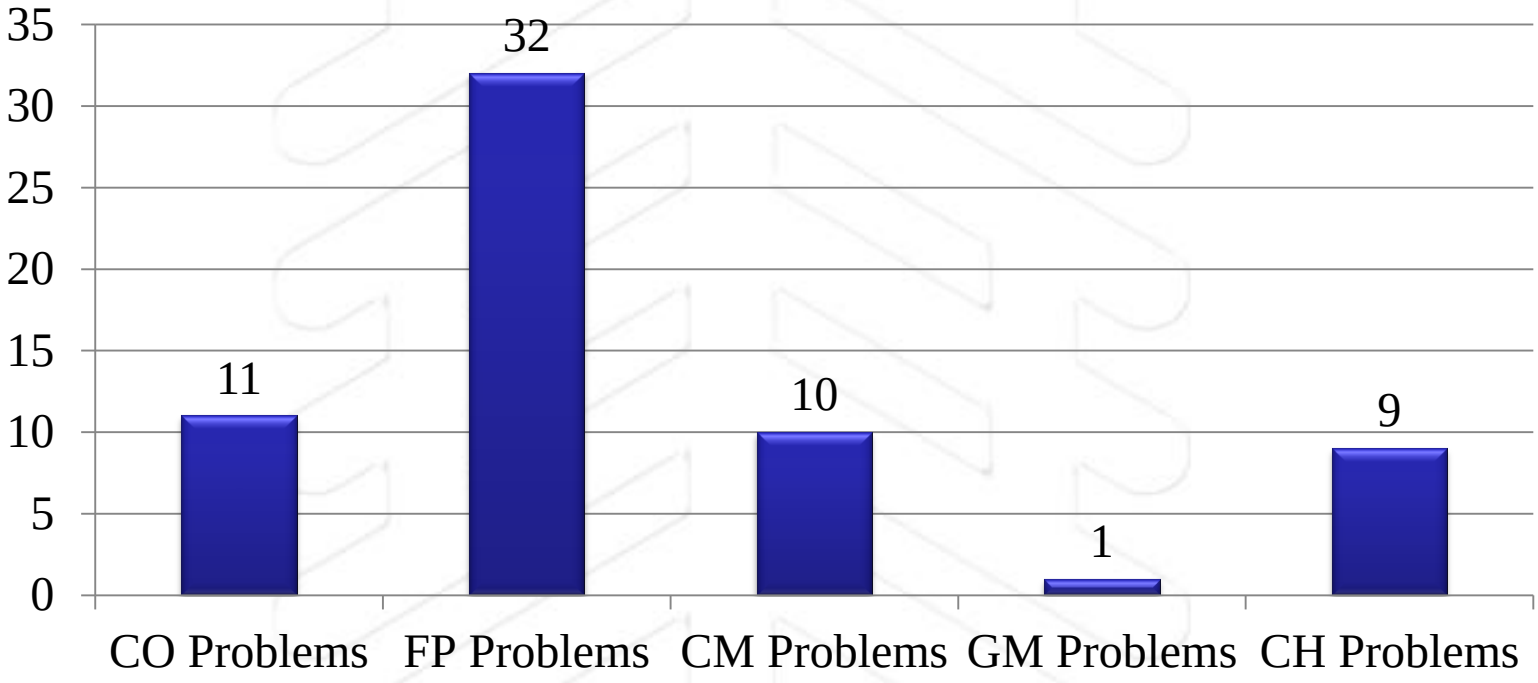
History-Based Problem Posing



Relationship between Approaches to Using History & Strategies of Posing History-Based Problems

History-Based Problems in 20 HPM Lessons

In all 20 HPM lessons, a total of 63 **history-based problems** are found.



Types of History-Based Problems in 20 HPM Lessons

Examples of CO problems

- **A problem from the old Babylonian Tablet: How long will it take a sum of money to double at 20 percent annually?**

HPM Lesson: ***The Concept of Logarithm*** **(10th grade)**

Use: ***Exchange between the exponent and the logarithm***

$$1.2^x = 2 \Leftrightarrow x = \log_{1.2} 2$$

Examples of CO problems

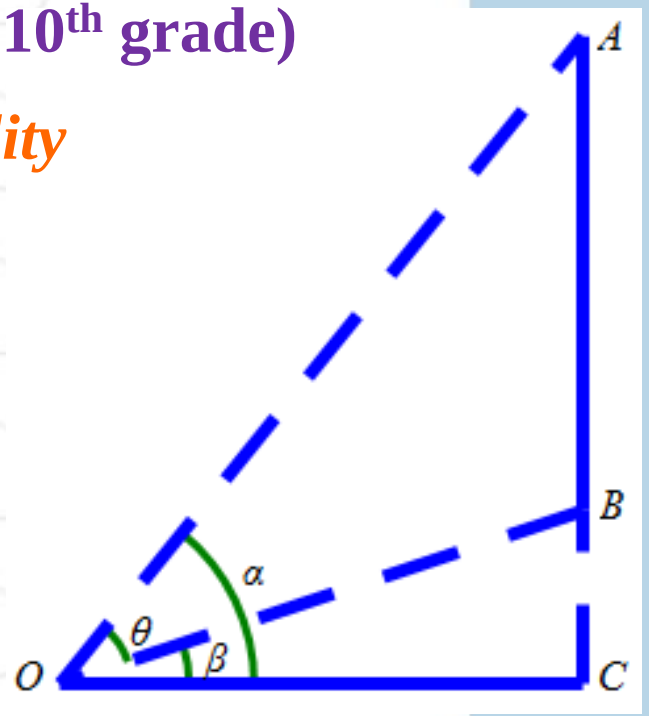
- **Regiomontanus' Angle Maximization Problem: At what point on the ground does a perpendicularly suspended rod appear longest?**

HPM Lesson: ***The Mean Value Inequality*** **(10th grade)**

Use: ***Application of the mean value inequality***

$$\tan\theta = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{a}{x}-\frac{b}{x}}{1+\frac{ab}{x^2}}$$

$$= \frac{a-b}{x+\frac{ab}{x}} \le \frac{a-b}{2\sqrt{ab}} \quad \left(x+\frac{ab}{x} \ge 2\sqrt{ab}\right)$$



Examples of CO problems

- **A problem baffling G. W. Leibniz & C. Huygens: Solve the system of quadratic equations with two unknowns: $x^2 + y^2 = xy = 2$.**

HPM Lesson: ***The Introduction of the Complex Numbers*** **(11^{th} grade)**

Use: ***Introduction of the imaginary numbers***

The sum of x and y is equal to the square root of 6, but neither of them is a real number. What are they?

$$\left(\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}\right)$$

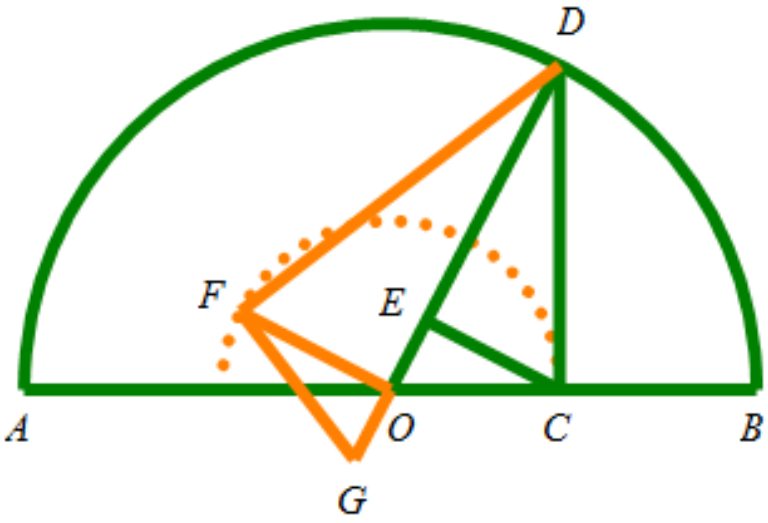
Examples of FP problems

- **The Greek mathematician Pappus constructed arithmetic, geometrical and harmonic means between segments *AC* = *a* and *CB* = *b*, as shown in the following figueCan you establish a chain of mean value inequalities?**

HPM Lesson: ***The Mean Value Inequality*** **(10th grade)**

Use: ***Derivation of the mean value inequality***

$$DE \le DC \le OD \Rightarrow \frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}$$



Examples of FP problems

- **Euclid defined the prism as follows: "A prism is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar and parallel, while the rest are parallelograms." Is this definition correct? Are there any counterexamples?**

HPM Lesson: ***The Definition of a Prism*** **(11th grade)**

Use: ***Revision & improvement of the definitions of a prism.***

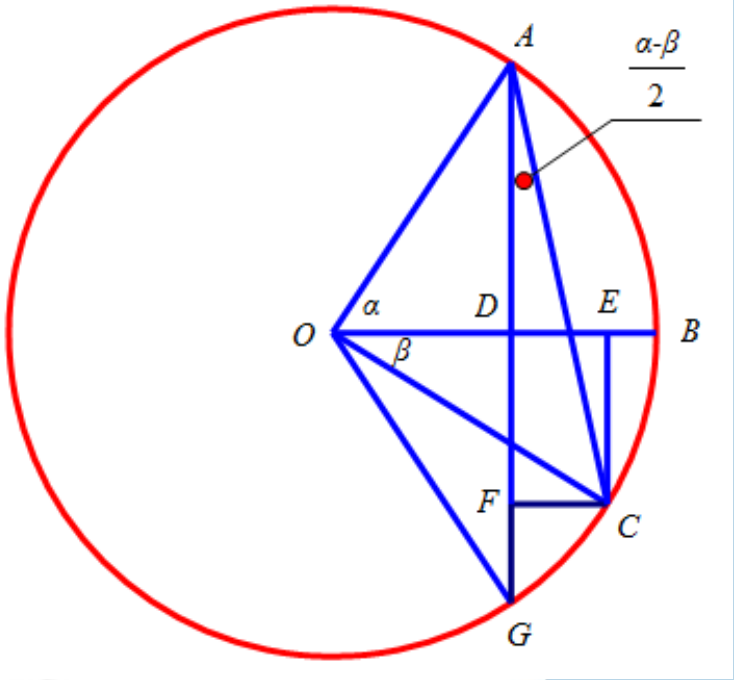
Before the 19th century, Euclid's definition was widely adopted by the geometry textbooks. The students also gave this definition. A counterexample appears in Stone & Millis (1916).

Examples of FP problems

- **The French mathematician F. Viète (1540-1603) derived the formula**

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

on the basis of the figure in the right.



Can you derived the formula

$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

with the same figure?

HPM Lesson: ***The Addition formulas for the Sine and Cosine*** **(11^{th} grade)**

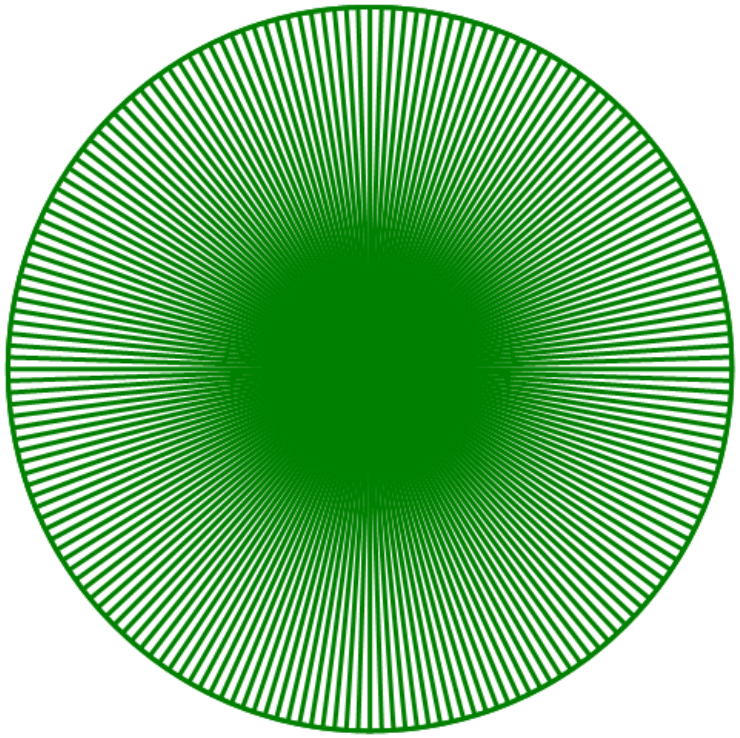
Use: ***Derivation of the trigonometrical formulas.***

Examples of FP problems

- **What is the idea underlying Liu Hui's Cyclotomic Method?**

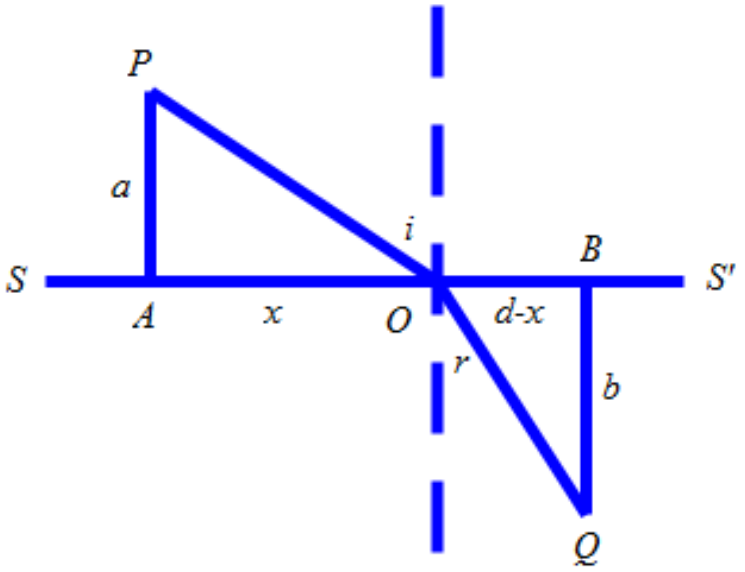
HPM Lesson: ***The Geometrical Meaning of the Derivative*** **(12th grade)**

Use*: Understanding the geometrical meaning of the derivative.*



Examples of CM problems

• Let two points *P* and *Q* be given in two media separated by the line *SS'* in the same plane. The velocities of the light in this two media are v_1 and v_2 respectively. Where is the point *O* on *SS'* when the light travels from *P* to *Q* and passes through *O*?



$$f(x)=\frac{\sqrt{a^2+x^2}}{v_1}+\frac{\sqrt{b^2+(d-x)^2}}{v_2}$$

$$\Rightarrow f'(x)=\frac{x}{\sqrt{a^2+x^2}}\frac{1}{v_1}-\frac{d-x}{\sqrt{b^2+(d-x)^2}}\frac{1}{v_2}$$

$$f'(x)=0$$

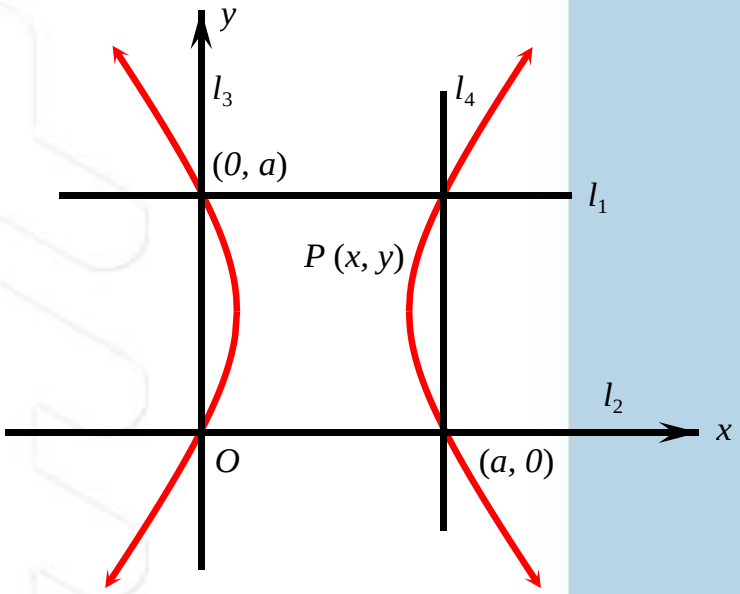
$$\Rightarrow \frac{\sin i}{\sin r}=\frac{v_1}{v_2}$$

HPM Lesson: ***The Application of the Derivatives*** **(12th grade)**

Use: ***Application of the derivatives***

Examples of CM problems

• $l_1 // l_2$, $l_3 \perp l_1$, $l_4 \perp l_1$, the distance between l_1 and l_2, l_3 and l_4 is a, the ratio of the product of the distances from P to l_1 and l_2 is to the product of the distances from P to l_3 and l_4 is as 2 to 1. What is the locus of P?



***HPM Lesson**: A Curve and its Equation* (11th grade)

***Use**: Application of the coordinate method*

Examples of GM problems

- **If one grain of corn is put in the first place and from the second place on, any place in the sequence of places is proposed to be double the sum of all preceding places. Find the general term of the sequence of grains of corn in each place of the chessboard.**

***HPM Lesson**: The Recurrent Sequences* **(11th grade)**

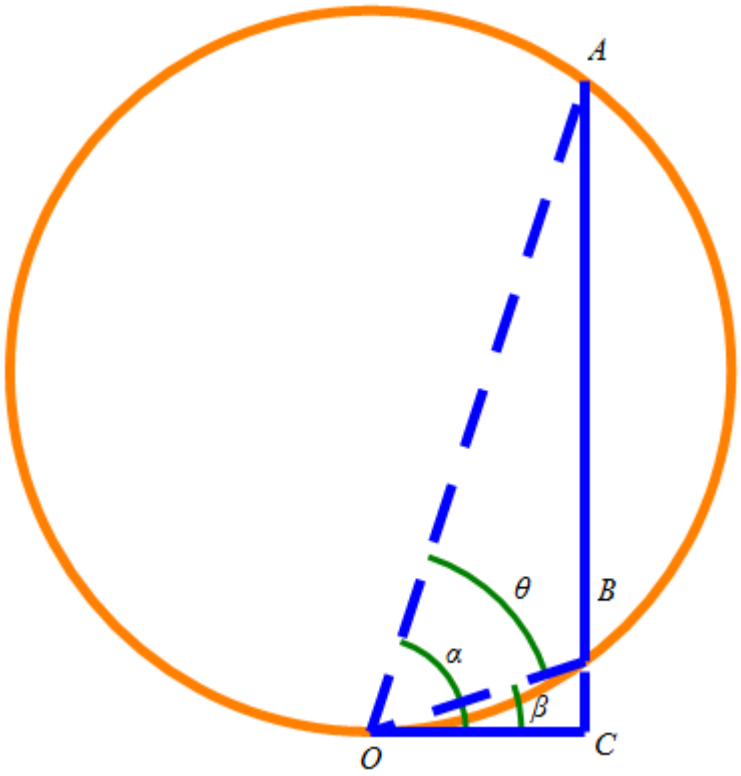
***Use**: The general term of the recurrent sequence*

Examples of CH problems

- **How to solve Regiomontanus's angle maximization problem geometrically?**

HPM Lesson: ***The Mean Value Inequality*** **(10th grade)**

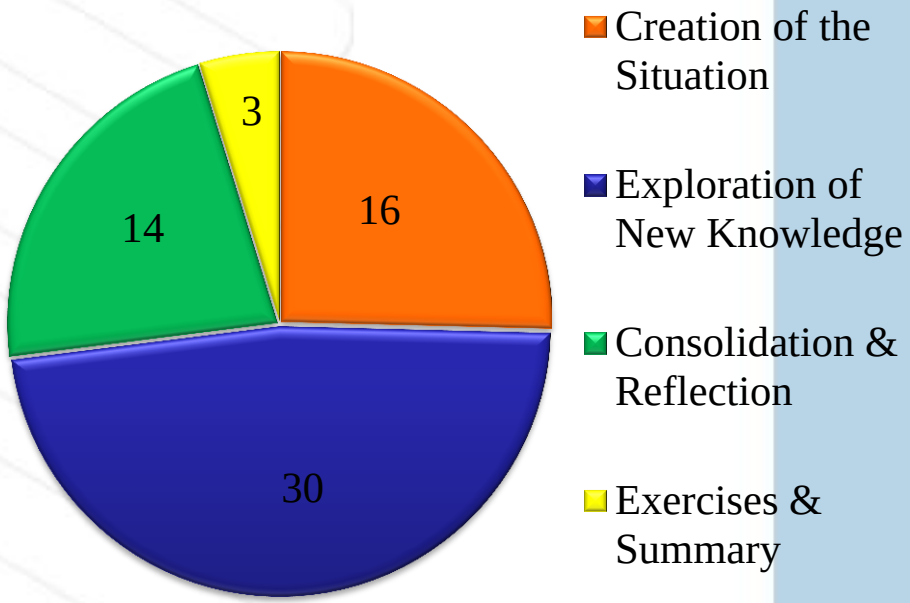
Use: ***The geometrical representation of the geometrical mean***



History-Based Problems in Different Parts of the Lessons

Five parts of a mathematics lesson:

- Review of the old know-ledge;
- Creation of the situations;
- Exploration of the new knowledge;
- Consolidation & reflection;
- In-class exercises & sum-mary



- **Core Mathematical competencies**

In mainland China, one of the general objectives of the revised senior high school mathematics curriculum is to develop students' core competencies (数学核心素养), including

- mathematical abstraction,

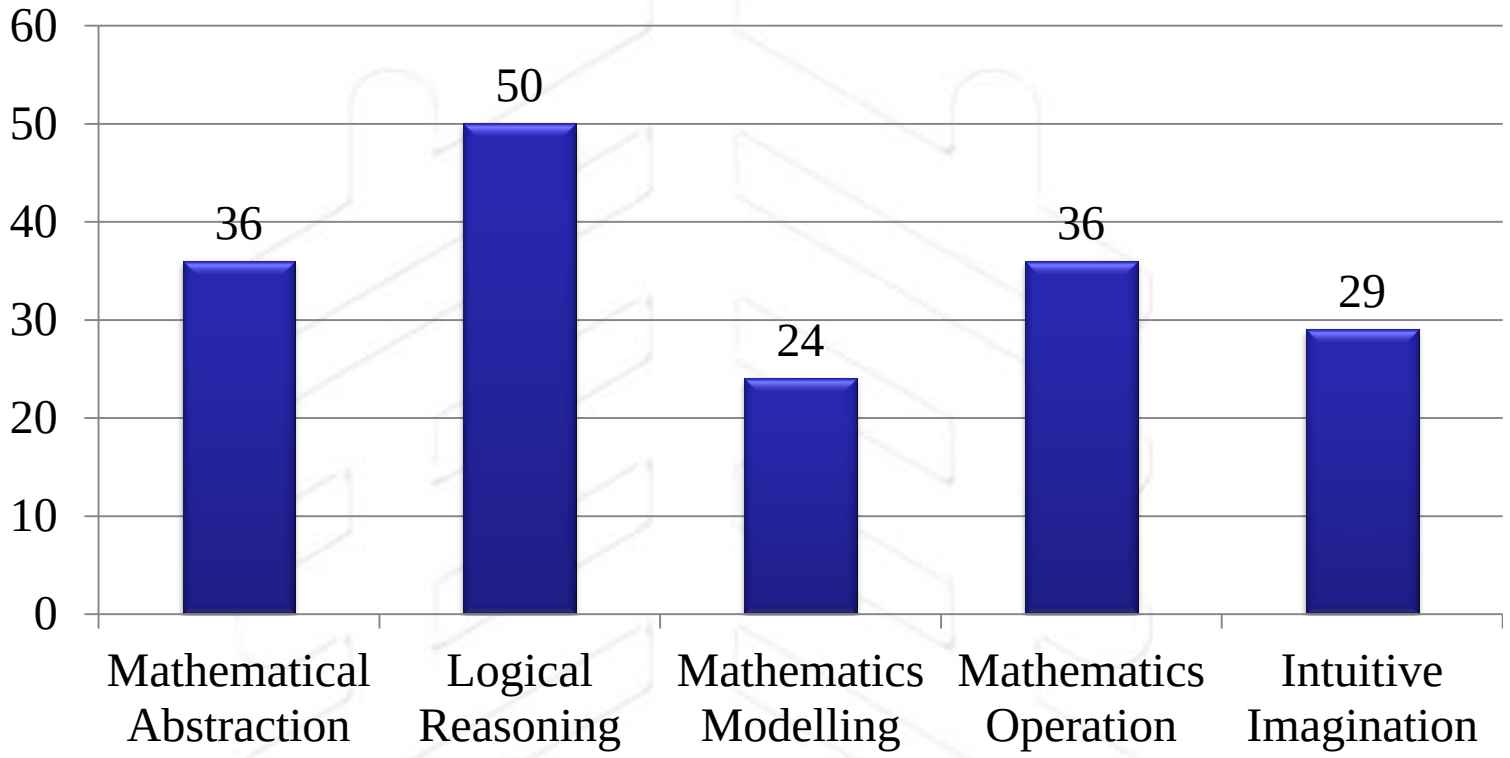
- logical reasoning,

- mathematical operation,

- Intuitive imagination

- data analysis.

Functions of History-Based Problems



Core Mathematics Competencies Embodied in Different History-Based Problems

- **Origin of Mathematical problems**

The history of mathematics offers us the **origin of problems**, from which endless new problems can be posed by means of various types of strategies. Moreover, history-based problems can easily form the **chains of problems**. For example, in the lesson “Curves and their Equations”, the two-line problem, three-line problem and four-line problem form an inseparable whole, creating for students the opportunity to experience the birth process of analytical geometry.

- **Affect and Beliefs**

History-based problems play unique roles in promoting positive affect and beliefs.

-In the lesson "**The Concept of Number Sequence**", the cat and mice problem of ancient Egypt and the Josephus' game aroused students' interest in number sequences.

-In the lesson "**The Concept of Complex Numbers**", Leibniz's system of equations with two unknowns stimulated students' learning motivation.

Conclusions & Implications

-In the lesson "**The Definition of Prism**", Euclid's incomplete definition of the prism told students that mathematicians might make mistakes and enabled them to recognize the nature of mathematics activity and reflect on their own mistakes and difficulties in learning mathematics.

-In the lesson "**The Application of Derivatives**", problems of can designing, dock locating, pipe installing and communication line connecting reveal the intimate links between mathematics and human real life, while the problem of deriving the refraction law shows that mathematics is closely related to other subjects.

Conclusions & Implications

- Of the 7 types of **history-based problems**, five are found in the 20 HPM lessons, which, according to the frequency of being used, **are FP, CO, CM, CH and GM problems** respectively.
- No SE problems or SM problems are found. Some problems are posed by means of situation manipulation, they are, however, classified among FP, CM or GM problems because the constraints or goals of original problems are also changed.

Conclusions & Implications

- History-based problems in 20 HPM lessons involves mathematics abstraction, mathematics modelling, logical reasoning, mathematical operations and intuitive imagination.
- In contrast with those non-historical problems, history-based problems had advantages in **affective and cultural aspects** of mathematics education.
- In 20 HPM lessons, history-based problems are mainly used for **creating situations and exploring knew knowledge** and seldom appear at other stages.

Conclusions & Implications

- **History-based problems** are indispensable for HPM lessons. Therefore, researchers should make deep and systematic historical researches on relevant subjects in the senior high school mathematics curriculum in an effort to amplify the repertoire of mathematical problems in mathematics curriculum, textbooks and teaching.
- Researchers should also cooperate with high school mathematics teachers, discussing and disseminating the history-based problem posing strategies.
- High school mathematics teachers should know more about the educational values of the history-based problems and using them at more stages of classroom teaching, completely exerting their educational functions.

Thank You for Your Attention

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Welcome to HPM-2020

(in China)