The Mathematical Cultures of Medieval Europe

Victor J. Katz
University of the District of Columbia (emeritus)
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Three mathematical cultures

- Arabic: Spain, beginning in 10^{th} century
- Hebrew: Spain, Provence, Italy, beginning in 11^{th} century
- Latin: Western Europe, beginning in 12^{th} century

Islamic Spain

- Conquest begins in 711
- Umayyad Dynasty, related to dynasties in Western Asia and North Africa, beginning in 750
- Caliphate under ʻAbd al- Raḥmān III (912-961) and al-Ḥakam II (961-977), capital in Cordova
- Caliphate ends in 1031 in reign of intellectual terror
- From 1031 into the 12th century, many small Islamic kingdoms in Spain
- Berber dynasties: Almovarids (1086-1145) and Almohads (1147-1238)

Catholic “Reconquista”

- Toledo (1085)
- Tudela and Zaragoza (1118)
- Lisbon (1147)
- Battle of Navas de Tolosa (1212)
- Cadiz and Cordoba (1236)
- Seville (1248)
- Granada (1492)

Jews in Spain

- Strong Jewish community under Umayyad Caliphate
- Some of the small Islamic kingdoms welcomed Jews, while others did not
- Berber dynasties often forced Jews to leave
- Catholic monarchs initially welcomed Jews, through 12th and early 13th centuries
- By 13th century, Jews were often persecuted by Catholic rulers and began to leave Spain, often for Provence
- Final expulsion of Jews in 1492

C'est à Montpellier...

Chassés d'Andalousie, des lettrés juifs s'installent dans le nord de l'Egypte ou dans le midi de la France. Ils poursuivent leurs échanges de part et d'autre de la Méditerranée. Dès lors, s'épanouit en Languedoc la culture andalouse d'expression arabe nourrie aux sciences antiques et grecques. Montpellier occupe une place centrale. La notoriété des intellectuels juifs ou « Sages du Mont » est considérable. Benjamin de Tudèle, rabbin voyageur venu de Navarre, décrit Montpellier comme la ville *« où exercent les plus grands lettrés de notre temps »*.
La ville est une oasis de tolérance et de dialogue : progrès dans la connaissance, dans l'accueil aux individus d'où qu'ils viennent, dans l'ouverture aux sciences d'où qu'elles proviennent.

Abū ʻAbdullah Muḥammad ibn ʻAbdūn (923-976)

- Lived in Cordova
- Physician to caliph al-Ḥakam II
- Surviving mathematical work is *On Measurement*
- Deals with rectangles, squares, triangles, parallelograms, circles (using 22/7 for π)
- Old method of solving quadratic equations (not using al-Khwārizmīan terminology of "thing" for the unknown and *māl* (treasure) for square).

Ibn ʻAbdūn problem

- If you are told, "We add the sides and the area and it is one hundred forty, what are the sides?" The calculation is that you add up the number of the sides, which is four, and take its half, two. Multiply it by itself, it is four. Add it to one hundred forty, which is one hundred forty-four. Then take the root of that, twelve, and take away from it half of the four and the remainder is equal to each of its sides.
- No justification given

Ibn Mu‘ādh al-Jayyānī (mid-11th c.)

- *Book of Unknowns of Arcs of the Sphere*
- In this book we want to find the magnitudes of arcs falling on the surface of the sphere and the angles of great arcs occurring on it as exactly as possible, in order to derive from it the greatest benefit towards understanding the science of celestial motions and towards the calculation of the phenomena in the cosmos resulting from the varying positions of celestial bodies. … As for premises that were derived by scholars who preceded us, we give just the statements, without proof, so that we may arrive at acknowledgement of their proof. …We have written our book for those who are already advanced in geometry, rather than for beginners.

Ibn Muʿādh al-Jayyānī

- We say that there are two kinds of things found in a triangle, sides and angles. There are three sides and three angles, but there is no way to know the triangle completely, i.e. [all] its sides and its angles, by knowing only two of the six. Rather, from knowing only two things, be they two sides or two angles or one side and one angle, it is unspecified. For it is possible that there are a number of triangles, each of which has those [same] two known things, and so one must know three things connected with it to obtain knowledge of the rest. Thus it is impossible to attain all of it knowing less than three members: three sides, three angles, two sides and an angle or two angles and a side.

Ibn al-Samḥ (984-1035)

- Lived in Cordova
- *The Plane Sections of a Cylinder and the Determination of their Areas*
- Book survives only in a Hebrew translation by Qalonymos ben Qalonymos of Provence.

Plane Sections of a Cylinder

- "Triangle of movement" is constructed by fixing one side of a triangle and moving the intersection of the other two sides in such a way that their sum is always equal, although the lengths of each will vary as their intersection moves.
- Oblique section of a right circular cylinder, already known to be an ellipse
- Shows that these two figures are identical

Plane Sections of a Cylinder

- The ratio of the inscribed circle to the ellipse is the same as the ratio of the minor to the major axis.
- The ratio of the inscribed circle to the ellipse is the same as the ratio of the ellipse to the circumscribed circle.
- Every ellipse is equal to the right triangle of which one of the sides containing the right angle is equal to the circumference of the inscribed circle and of which the second side is equal to half of the greatest diameter. It results from what we have established that if we take five sevenths and one half of one seventh of the smallest diameter, and multiply this by the greatest diameter, we obtain the area of the ellipse.

Al-Mu'taman Ibn Hūd (d. 1085)

- Member of ruling dynasty of Saragossa and served as king from 1081 to 1085.
- *Kitāb al-Istikmāl* (*Book of Perfection*) was an extensive survey of mathematics, including ideas from Greek and Arabic sources and probably some of his own contributions as well.
- Gave a new proof of Heron's Theorem
- Detailed study of "Alhazen's Problem" concerning reflection in mirrors whose surfaces are curved. Based on work of ibn al-Haytham, but with different, and often better, proofs

Book of Perfection

- Statement and proof of a theorem thought to have been originated by the Italian geometer Giovanni Ceva in 1678.
- [Ceva's Theorem:] In every triangle in which from each of its angles a line issues to intersect the opposite side, such that the three lines meet inside the triangle at one point, the ratio of one of the parts of a side of the triangle to the other [part], doubled with the ratio of the part [of the side] adjacent to the second term [of the first ratio] to the other part of that side is as the ratio of the two parts of the remaining side of the triangle, if [this last] ratio is inverted, and conversely.

Spanish Muslim mathematics

- How did a mathematician work in Muslim Spain?
- Either he had another career to provide support, or else, he was supported by – or in the case of ibn Hūd, actually was – a ruler of the state in which they lived.
- No structure existed in this society that could support a steady flow of intellectual development, such as a university. If one wanted to study advanced mathematics, one had to go to Egypt or Persia or Baghdad.
- Mathematicians restricted themselves to certain topics, in particular, geometry and trigonometry, both based on a firm knowledge of Euclid's *Elements*.
- Muslim mathematicians had also read Archimedes, Apollonius, and Ptolemy, among other Greek authors. They certainly absorbed the Greek notion of mathematical proof.

Spanish Muslim mathematics

- Sā‘id al-Andalusī , who wrote *Science in the medieval world: Book of the categories of nations* in 1068 names many other mathematicians active in al-Andalus, but their fields of interest were geometry and astronomy.
- No algebra more advanced than al-Khwārizmī studied in Spain.
- Averroes (1126-1198) translated and commented extensively on the work of Aristotle, but Muslim mathematicians did not attempt to develop the mathematics implied by Aristotle's physical ideas.
- Little evidence in Spain that there were any religious restrictions to the practice of mathematics. So the reasons why one topic was studied had to do with practical reasons, such as the availability of teachers, or, more simply, with the inclinations of a particular mathematician.

Averroès (1126-1198)

Foi et raison dans le monde arabe et chrétien

Abu l-Walid Muhammad ibn Rushd est un savant andalou, né à Cordoue et mort à Marrakech, connu aussi sous le nom latinisé d'Averroès. Comme théologien, il se situe dans le cadre d'une réforme rationalisante de l'Islam. En philosophie, il est avant tout « le commentateur » d'Aristote. L'œuvre d'Averroès – en traduction latine ou hébraïque a été conservée par le courant rationaliste du judaïsme d'Europe. Semblablement dans le monde chrétien, la *Somme théologique* de Saint Thomas d'Aquin (1125-1274) s'organise autour du thème central d'une harmonie entre la foi et la raison.

Abraham bar Ḥiyya (1065-1145)

- Community leader (*nasi*) in Barcelona
- Head of the guard (*ṣāḥib ash-shurṭa*, savasorda), title given from ibn Hūd dynasty in Saragossa
- First Jewish scholar in Arabic-speaking world to write on science in Hebrew, including astronomy, philosophy, astrology
- Mathematical work was *The Treatise of Measuring Areas and Volumes*, partially translated into Latin in 1145 by Plato of Tivoli; almost certainly read by Fibonacci, among others.

Treatise of Measuring Areas and Volumes

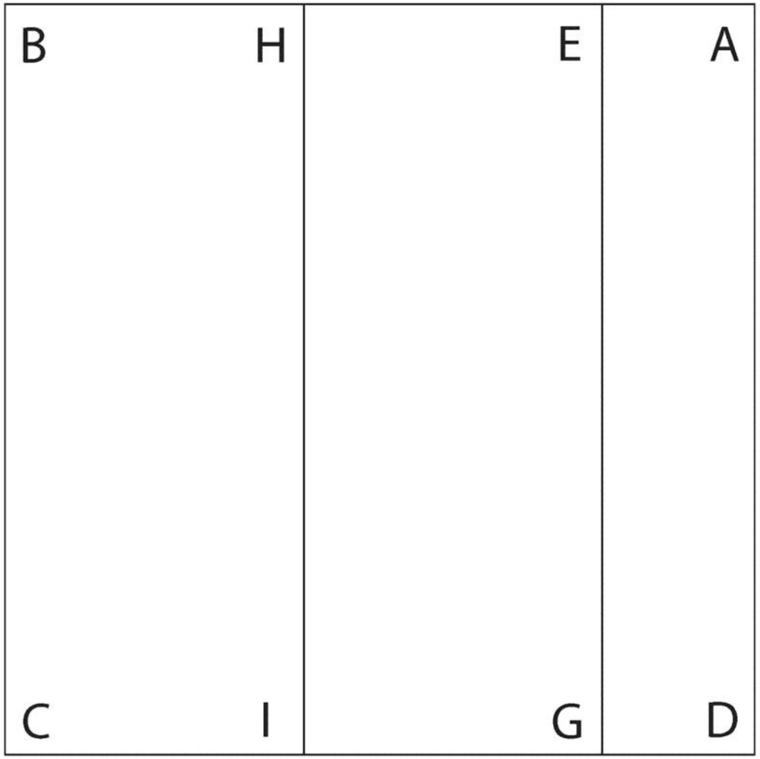
- [The scriptures say] “I the Lord am your God, instructing you for your own benefit, guiding you in the way you should go”, that is, instructing you in whatever is useful for you, and guiding you on the way you follow, the way of the *Torah*. From which you learn that any craft and branch of wisdom that benefit man in worldly and holy matters are worthy of being studied and practiced.
- I have seen that arithmetic and geometry are such branches of wisdom, and are useful for many tasks involved in the laws and commandments of the *Torah*.

Treatise of Measuring Areas and Volumes

- No man can calculate precisely without falsification unless he learn arithmetic. But he who has no knowledge and practice in geometry cannot measure and divide land truly and justly without falsification.
- Arithmetic is not difficult to understand
- Geometry is also as useful for as many matters as arithmetic in worldly matters and commandments from the *Torah*, but is difficult to understand, and is puzzling to most people, so one has to study and interpret it for land measurement and division between heirs and partners, so much so that no one can measure and divide land rightfully and truthfully unless they depend on this wisdom.
- Our fathers did not allow us to dismiss calculations, nor steal from heirs, nor give any of them more or less than their fair share…. They warned us and gave us strict orders against stealing and falsifying in measuring land …

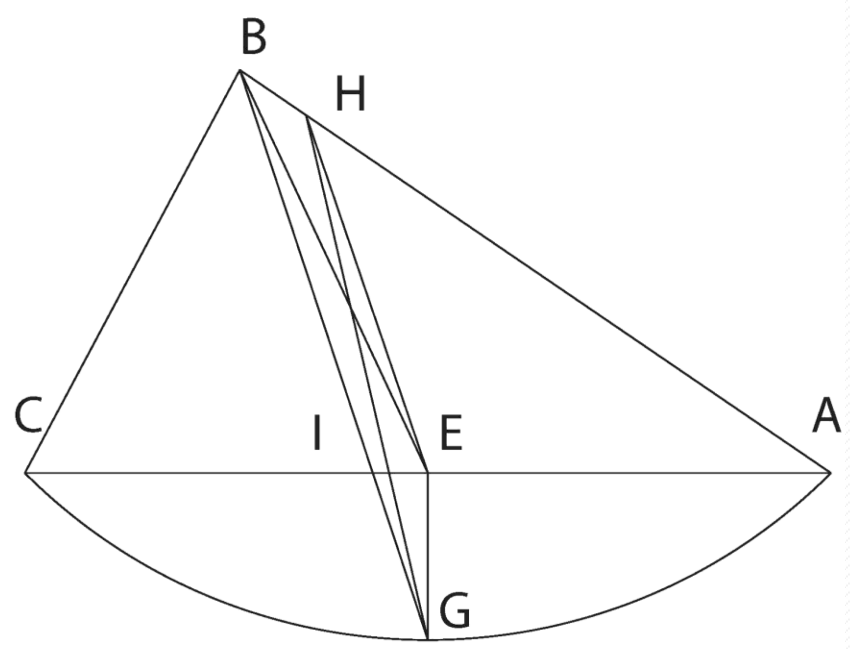
Treatise of Measuring Areas and Volumes

- A square quadrilateral that you take away from the number of its area the number of its four sides, and are left with 21 cubits of its area: what is the area and what is the number of each side of the square?
- Standard algorithm with proof using diagram and *Elements* II-6



Treatise of Measuring Areas and Volumes

- To find a straight line dividing the region in half. *E* is the midpoint of *AC* and *EG* is perpendicular to *AC*. *BG* intersects *AC* at *I*; draw *HE* parallel to *BG*. Then *GH* divides the region in half.
- Proof: *BE* divides triangle *ABC* in half, while *EG* divides segment *AECG* in half. But triangles *GHE* and *BHE* are equal. So region *AHG* is equal to the sum of triangle *ABE* and region *AEG*, so is half of the entire region *ABCG*.



Abraham ibn Ezra (1090-1167)

- Born in kingdom of Saragossa, but traveled widely throughout his life.
- Wrote books on arithmetic, numerology, astrology, as well as Biblical commentaries
- Only when one knows the natural sciences and their proofs, learns the categories that are the 'guardians of the walls' taught by the science of logic, masters the science of astronomy with its absolute proofs based on mathematical knowledge, and comprehends the science of geometry and the science of proportions, can one ascend to the great level of knowing the secret of the soul, the secret of the supernal angels, and the concept of the world to come in the Torah, the Prophets, and the sages of the Talmud.

Abraham ibn Ezra

- Reuven hired Simon to carry on his beast of burden 13 measures of wheat over 17 miles for a payment of 19 *pashuts*. He carried seven measures over 11 miles. How much shall be paid?
- There are 120 conjunctions [of the seven planets]. You can calculate their number in the following manner: it is known that you can calculate the number that is the sum [of all the whole numbers] from one to any other number you wish by multiplying this number by [the sum of] half its value plus one-half.
- $C_{7,3} = C_{6,2} + C_{5,2} + C_{4,2} + C_{3,2} + C_{2,2}$

Abraham ibn Ezra

- If one is dealing with a semicircle, its area is like that of half a circle. It it is smaller or larger, you must know the diameter of the circle from which the circular segment has been cut, and the length of the chord of the arc and of the sagitta. When you know two of these elements, you can determine the third. Problem: The chord is 8, the diameter, 10. How much is the sagitta? Subtract from the square of half the diameter the square of half the chord; take the root of the remainder, and subtract it from half the diameter; you will find the sagitta [= 2]

Bahya ibn Paquda (mid 11th c.)

- All departments of science are gates which the Creator has opened to rational beings, so that they may attain to a comprehension of revealed religion and of the world. But while some sciences satisfy primarily the needs of religion, others are more requisite for the benefit of the world. The sciences specially required for the affairs of the world are the science that deals with the natures and accidental properties of physical substances – and the science of mathematics. These two branches of knowledge afford instruction concerning the secrets of the physical world and the uses and benefits to be derived from it, as well as concerning arts and artifices needed for physical and material well-being. But the science that is needed primarily for revealed religion is the highest science, namely the divine science, which we are under obligation to study in order to understand our revealed religion and to reach up to it. To study it, however, for the sake of worldly advantages is forbidden to us.

Maimonides (1135-1204)

- It is certainly necessary for whoever wishes to achieve human perfection to train himself first in the art of logic, then in the mathematical sciences according to the proper order, then in the natural sciences, and after that in the divine science.
- So certainly the study of Euclid's *Elements* was legitimate as was trigonometry. But somehow, the study of algebra was pointless, as it was a mere technique, having no philosophical value or even practical use.

Abraham ibn Daud of Toledo (1110-1180)

- Among those who spend their time on vanities, thereby depriving their soul of afterlife is he who consumes his time with number and with strange stories like the following: A man wanted to boil fifteen quarters of new wine so that it be reduced to a third. He boiled it until a quarter thereof departed, whereupon two quarters of the remaining wine were spilled; he again boiled it until a quarter vanished in the fire, whereupon two quarters of the rest were spilled. What is the proportion between the quantity obtained and the quantity sought?

Levi ben Gershon (1288-1344)

Maasei Hoshev (*The Art of the Calculator*)

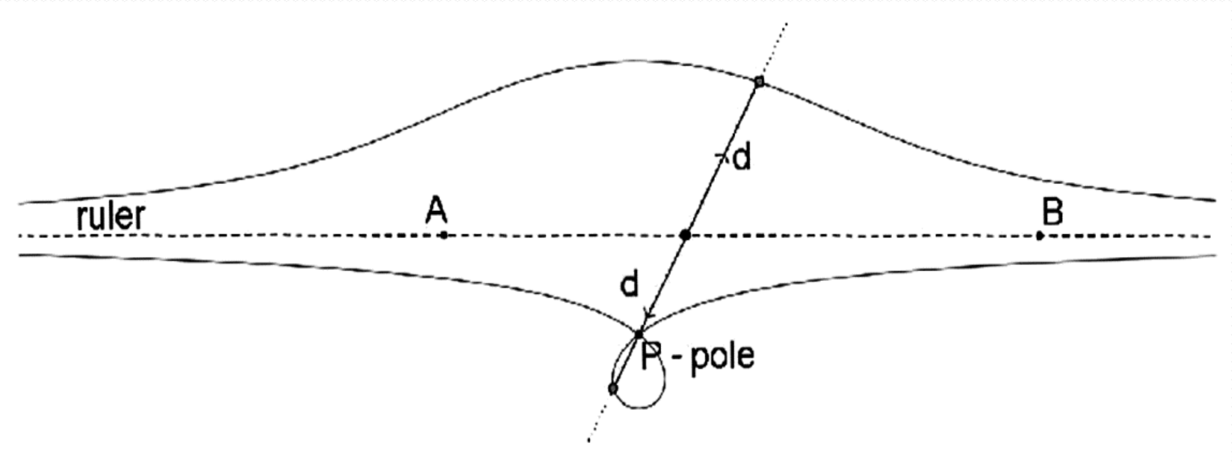
- When you are given a number of terms and the number of permutations of a second given number from these terms is a third given number, then the number of permutations of the number following the second given number from these terms is the product of the given third number by the excess of the first given number over the second number.
- In modern terms, this result is $P_{m,n+1} = (m - n)P_{m,n}$.
- This result is the inductive step for proving that $P_{m,k} = m(m - 1)(m - 2) \ldots (m - k + 1)$, a theorem Levi states next.

Levi ben Gershon

- Commentary on Euclid's *Elements*
- The straight line which is inclined [to another straight line] approaches [the second line] on the side where an acute angle is formed [with a line crossing both of these that is a perpendicular from the first line to the second].
- *On Harmonic Numbers*
- A power of 2 must differ from a power of 3 by at least 2, except in the cases 1,2; 2,3; 3,4; and 8,9.

Abner of Burgos (1270-1348)

- Lived in Castile and converted to Christianity
- *Book of the Rectifying of the Curved*
- Defines and uses the conchoid of Nicomedes
- Locus of all points lying at distance *d* from the ruler *AB* along the segment that connects them to the pole *P*



Abner of Burgos

- Trisection of an angle
- Construction of two mean proportionals between two given line segments
- Construction of a polyhedron equal in volume to a given polyhedron and similar to a second given polyhedron. This is a special case of the Delian problem, since one can assume the first polyhedron is any parallelepiped of volume 2, while the second is a cube of volume 1.

Algebra in Hebrew

- Isaac ibn al-Ahdab (1350-1430); born in Castile but lived in Sicily. Wrote commentary on work of Ahmad ibn al-Banna'.
- Simon Motot (mid 15th c.); lived in Italy and learned his algebra in the Italian abacus tradition of the time.

Medieval Hebrew mathematics

- Levi ben Gershon decided that he could study and write on any topic he thought interesting, but few followers.
- Conflict within the Jewish community regarding what subjects could legitimately be studied.
- No institutional infrastructure for new students to learn the works of their predecessors. One could always arrange to study privately with an individual, and certainly there were "study groups" established by various people.
- No Jewish universities – just as there were no Muslim universities.

Leon Joseph of Carcassonne (c. 1400)

- Many years ago I directed my attention toward the study of and research into the profane sciences, which are several in number and nature.
- Sciences defeated them because their subject matter is more rational than in the bosom of our people, and they are as far from them as east is from west, and all the more so from the fundamentals of the Torah and of religious faith.
- [Those few who did study the sciences] had no right to propound [their knowledge] in the squares and streets, or to discuss it, to show themselves to be favorable toward it, nor to conduct public debates with the aim of reading the complete truth, for knowledge of the truth can only be attained by means of the contrary.

Des dynasties de traducteurs

Les familles de savants méridionaux entretiennent la pratique de l'arabe, langue des lettrés ibériques, qui permet d'aborder les œuvres fondamentales produites en Espagne. Les savants juifs traduisent des ouvrages de l'arabe en hébreu. Au sein de la famille exceptionnelle des Tibbonides, s'opère une vaste entreprise de traduction. Moïse Ibn Tibbon transpose en hébreu de multiples travaux scientifiques et médicaux venus de l'arabe : des Persans Rhazès et Avicenne ou encore de l'Andalou Averroès (contemporain de Maïmonide), médecins et philosophes inspirés par la philosophie grecque.

Leonardo of Pisa (1170-1240)

- *Liber Abbaci* (1202)
- There is a tree 1/4 + 1/3 of which lies underground, and it is 21 palms. It is sought what is the length of the tree.
- Two men who had *denari* found a purse with *denari* in it; thus found, the first man said to the second, If I take these *denari* of the purse, then with the *denari* I have, I shall have three times as many as you have. Alternately, the other man responded, And if I shall have the *denari* of the purse with my *denari*, then I shall have four times as many as you have.

Fibonacci

- *Practical Geometry*
- Measurement problems, some involving quadratic equations
- Problems on dividing region into two equal parts, some similar to those found in work of bar Ḥiyya.
- *Book of Squares*
- Find a square number from which when five is added or subtracted always arises a square number.

Universities of Medieval Europe

- University of Paris
- Oxford University
- Cambridge University
- University of Bologna
- University of Montpellier
- Curriculum based on trivium of logic, grammar, rhetoric along with quadrivium of arithmetic, geometry, music, astronomy.
- Much of curriculum based on logical work of Aristotle

More on Universities

- Universities were corporate bodies operating under royal charter and independent of church control
- Aristotle's philosophy sometimes posed problem for Catholic theologians
- In 1277, Bishop of Paris drew up a list of 210 "errors" of some scholars at Paris
- But the condemnation was generally ignored
- A new canon law had been developed in the 12th century stating that "anyone (and not just priests) ought to learn profane knowledge not just for pleasure but for instruction, in order that what is found therein may be turned to the use of sacred learning."

Oxford calculators (Merton College)

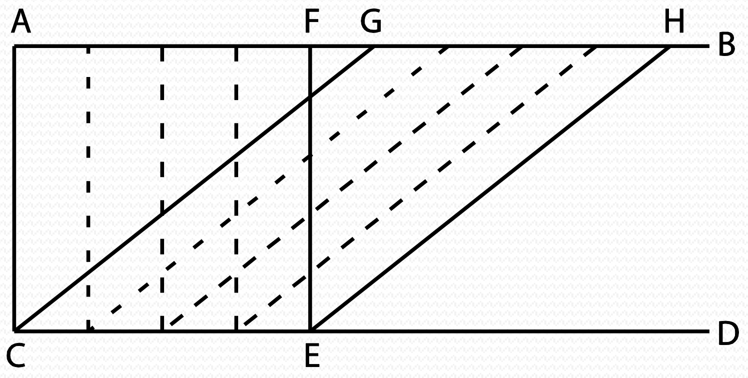
- Applied logical argument to Aristotle's physical problems
- Thomas Bradwardine (1290-1349), *On the Continuum*
- One must know that the old and modern philosophers have five famous opinions about the composition of the continuum. Some of them argue that the continuum is not composed of atoms, but of parts that can be divided without end. Others say that it is composed of two kinds of indivisibles. Others say that it consists of points, and this [assumption is divided] into two parts: either a finite number of indivisibles, or an infinite number. This group, too, is divided into two parts. Some say that it is composed of an infinite number of indivisibles that are directly joined. But others say that it is composed of an infinite number that are indirectly joined to one another. Therefore, if one continuum is composed of indivisibles in some way, it follows that any continuum is composed of indivisibles according to a similar way.

Thomas Bradwardine

- Rejects assumption that continuum is composed of a finite number of points:
- If this is true, then half of the circumference is equal to its diameter. From the different points of the diameter, assuming that they are 10, ten perpendiculars are drawn directly to different points on half the circumference. It follows that there are 10 points on half the circumference, because only one point on half the circumference corresponds to a perpendicular. Therefore equally, there are the same number of points on half the circumference as are on the diameter. Therefore, half the circumference equals the diameter.

Thomas Bradwardine

- Rejects hypothesis that continuum is composed of an infinite number of indivisibles.
- All lines of *CGHE* drawn from all points of *CE* to the opposite points of *GH* are equal in number to those points, and consequently to all perpendiculars of *AFCE* drawn from the same points to the opposite points. But they are longer than those lines. So *CGHE* is larger than *AFCE*. But this contradicts *Elements* I 36.



Bradwardine's Conclusion

- No continuum is made up of atoms. From here follows and elicits: Every continuum is composed of an infinite number of continua of the same species as it, that is, every line is composed of an infinite number of lines, every surface composed of an infinite number of surfaces, and so on concerning other continua.

Bradwardine *Treatise on Proportions*

- Relationship among speed (*V*), force (*F*), and resistance (*R*).
- Now that these fogs of ignorance, these winds of demonstration, have been put to flight, it remains for the light of knowledge and of truth to shine forth.
- Theorem I. The proportion of the speeds of motions follows the proportion of the force of the mover to that of the moved, and conversely. Or, the proportion of motive to resistive power is equal to the proportion of their respective speeds of motion, and conversely. This is to be understood in the sense of geometric proportionality.
- $V = \log_n (F/R)$ or as $n^V = F/R$. That is to say, doubling the velocity squares the ratio of motive power to resistance, tripling the one cubes the other, and so on.

William Heytesbury (1313-1373)

- Mean Speed Theorem: A body that moves with uniformly accelerating speed traverses in a given time the same distance as a body that in the same time moves with a constant speed equal to the accelerating body's speed at the middle instant.
- Corollary: Under uniformly accelerated motion from rest, a body in the first half of a given interval will traverse one-third of the distance it covers in the second half of the interval.

Nicole Oresme (1320-1382)

- *On the Configurations of Qualities and Motions*
- Every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible thing, e.g., a quality. For whatever ratio is found to exist between intensity and intensity, in relating intensities of the same kind, a similar ratio is found to exist between line and line, and vice versa.

Nicole Oresme

- For curvature, like the other qualities, has both extension and intensity, and one kind of curvature is uniform while another is difform. But still it is not manifest, in regard to the ratio of the intensity of curvatures, whether one is double another or exists in another ratio to it, or whether or not curvatures are unrelatable one to the other by ratio.
- Every circular curvature is uniform and vice versa, and every other curvature is difform. The intensity of circular curvature is measured by the quantity of the radius of the circle whose curve is… the circumference, so that by the amount the radius is less, so proportionally the curvature will be greater.

Abbacus mathematics in Montpellier

- Jacobo da Firenze, *Tractatus algorismi* (1307)
- Paolo Girardi, 1327
- A man loaned 20 *lire* to another for two years at compound interest. When the end of 2 years came he gave me 30 *lire*. I ask you at what rate was the *lire* loaned per month?
- There is a man who went on 2 voyages. On the first voyage he earned 12 *denarii*. On the second voyage he earned at the same rate that he made on the first voyage, and at the end he found [he had] 100 *denarii*. I ask you with how many *denarii* did he leave?

Comparisons

- Common mathematical background included Hindu-Arabic number system, works of Aristotle, and Euclid's *Elements*.
- Muslims: No algebra beyond al-Khwārizmī. Definite interest in geometry, both practical and theoretical, with obvious use of Greek techniques in the latter. Developed trigonometry, especially spherical, for use in astronomy. Important for religious purposes.
- Jews: Not interested in algebra, although used older methodology of manipulation of geometric figures in solving quadratic problems in the context of measuring. Definitely interested in advanced geometry, combinatorics, and even some number theory. Trigonometry was also studied for astronomical and calendrical purposes.

Comparisons (cont)

- Catholics more interested in developing algebra beyond al-Khwārizmī, although little of eastern Islamic algebra reached Europe during medieval period. Little interest in advanced geometry or any new development of trigonometry until mid-fifteenth century. Little interest in combinatorics. Most important mathematical topic was the set of developments stemming from study of Aristotle's physical theories, especially the application to kinematics.