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MATHEMATICS & PHYSICS: AN INNERMOST RELATIONSHIP
Didactical implications for their teaching & learning

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Teaching & learning
mathematics & **physics** kept **separated**

A ***distinction*** *at the* ***research level***:

- mathematicians stay in a universe of ideal **logical rigor**
- physicists are **users** of mathematics

Reflected *in*

- physics education: mathematics is a **tool** to describe and calculate,

-mathematics education: physics is a **domain of application** of mathematics previously conceived abstractly

To overcome this ***dichotomy***:

Systematic research in different fields, on historical, philosophical and sociological aspects of scientific knowledge

History does **not verify** this **separation**

less than 100 year-old!
characterizing development up to the 1960-70s at the peak of the effort to extremely formalize mathematics (reflected in ME in the "New Math" reform)

- **No** clear-cut **separation before**
- Recently strong **tendency** to **overcome** it

E.g.

- *"This grand book, the Universe… is written in the language of Mathematics"* (Galileo 1623)
- Hilbert's 6th problem (Hilbert 1900) *Mathematical treatment of the axioms of physics*
- *"The unreasonable effectiveness of mathematics in the natural sciences"* (Wigner 1960)

History in **Mathematics** & **Physics Education**:

promising way to **teach & learn**

Mathematics & Physics

In principle, it provides the opportunity **to appreciate** the **evolutionary nature** of **scientific knowledge**

A worldwide intensively studied area of new pedagogical practices & research activities

Structure of this talk

Part 1: **Historical – Epistemological framework**

Three main **theses**

Thesis A: The **ontological status** - What is Mathematics? What is Physics?

Thesis B: The **interrelated historical development** of Mathematics & Physics

Thesis C: The **epistemological affinity** of Mathematics & Physics

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

- ***Which*** **history** for didactical purposes?
- With ***which role(s)***?
- In ***which way(s)*** to be realized in practice?

Part 3: **Illustrative Examples**

1. Measuring the distance of inaccessible objects
2. Rotations, Space-Time & Special Theory of Relativity
3. Differential Equations, (Functional) Analysis & Quantum Mechanics

Part 1: **Historical – Epistemological framework**

Thesis A: The **ontological status** - What is Mathematics? What is Physics?

Mathematics & Physics should be conceived - hence, taught and learnt - both as

- ***result*** of intellectual enterprises
- ***procedures*** leading to these results

Knowledge in their context has an **evolutionary character**; by its very nature,

historicity is a deeply-rooted characteristic

Perceiving mathematics or physics **both** as a

- *Logically structured collection of intellectual products* **and**
- *Knowledge-producing endeavours*

should be the core of their teaching and central to their image communicated outside

Implication for Education:

Historical & Epistemological issues in teaching & learning **Mathematics** & **Physics**: a possible **natural way for exposing them in the making**, leading to

- better **understanding** their specific parts,
- **deeper awareness** of what they are as disciplines

Thesis B: The **interrelated historical development** of Mathematics & Physics

From **antiquity** to the **present**, **Mathematics** & **Physics** evolve in close, continuous, uninterrupted, bidirectional, multifaceted, fruitful way

- Hero's geometrical proof of the *law of reflection*
- Eratosthenes' estimation of the *earth's circumference*
- Archimedes' "mechanical arguments" in his *Method*
- Poincaré's group theoretic derivation of the *Lorentz transformations* in SR
- Hilbert's deduction of *General Relativity field equations* from a variational principle
- von Neumann's rigorous formulation of QM
- Penrose's *singularity theorems* in General Relativity
- Feynman's *path-integrals* in Quantum Mechanics & *functional integration*
- Thom's *Catastrophe Theory*
- Connes' *Non-commutative Geometry* & Quantum Field Theory

A **simplified** scheme of **3 scenarios**

(S_1) ***Parallel*** *development*: Physical problems asking for solution & formulation of appropriate mathematics (concepts, methods, or theories) evolve in parallel

- Infinitesimal calculus, Classical Mechanics (17^{th} century)
- Vector Analysis, Electromagnetism, Fluid Mechanics (19^{th} century)
- Statistical concepts, error theory in Celestial Mechanics, Kinetic Theory (19^{th} century)
- 1^{st} order PDE, Geometrical Optics, Classical Mechanics (Hamilton)

(S_2) ***Mathematical*** *concepts, methods or theories* ***precede*** *their integration into* ***physics***: The corresponding physical problems naturally stress the need for the appropriate mathematics

- Riemannian Geometry, Tensor Calculus, General Relativity
- Matrix Algebra, Matrix Mechanics (Heisenberg)

(S_3) ***Physical*** *problems* ***precede*** *the formulation of* ***mathematics*** *appropriate to tackle them*: Partially intuitive, formal or experimentally-induced models, and logically incomplete, ill-defined concepts, motivate and/or guide the development of new mathematics

- Brownian motion (Langevin), Stochastic differential equations,
- Dirac's δ-function, Distribution theory
- Path integrals (Feynman), Functional integration

Thesis C: The **epistemological affinity** of Mathematics & Physics

C(a): Mathematics & Physics always closely interwoven
A **bi-directional** process:

- ***From* Mathematics *to* Physics**:

  Mathematics is the ***language of physics***,

- **not only** as a **tool** for expressing, handling and developing logically physical concepts, methods and theories,
- **but also** as an indispensable, **formative characteristic** **shaping** them, by deepening, sharpening, and extending their meaning, or even **endowing** them with **meaning**

- ***From* Physics *to* Mathematics**:

  Physics is a natural **framework**

- **not only** for testing, applying and elaborating **mathematical** theories, methods and concepts,
- **but also** for motivating, stimulating, instigating, **creating** all kinds of **mathematical innovations**

- Maxwell (1856): *Natural philosophy is, and ought to be, Mathematics… the greatest advances in mathematics have been due to enquirers into physical laws*

- Weyl (1922): *Geometry, Mechanics, and Physics form an **inseparable theoretical whole***

- Einstein (1934): ***Experience** can… **guide** us in our choice of serviceable mathematical concepts…* [and] *remains the sole **criterion** of the serviceablility of a mathematical construction for physics, but the truly **creative principle** resides in **mathematics***

- Wigner (1960): the unreasonable effectiveness of **mathematics** in the natural sciences "*…shows that* [mathematics] *is in a **very real sense**, the **correct language**…*" and its predictions, often in amazing agreement with experiments indicate that "*surely… we 'got something out' of the equations that we **did not put in***"

- Dirac (1979): *Anyone who appreciates the fundamental **harmony** connecting the way **nature** runs, **and** general **mathematical** principles, must feel that a theory with… beauty and elegance… **has to be** substantially correct*

- Hilbert (1902):

" … while the creative power of pure reason is at work, the outer world … comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus … advance most successfully the old theories. … the numerous and surprising analogies and ***that apparently prearranged harmony*** *which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science,* ***have their origin*** *in this* ***ever-recurring interplay between thought and experience*** *…"*

Thesis C: The **epistemological affinity** of Mathematics & Physics

C(b): Mathematics & Physics as **embodiments of general attitudes in regard to the description, exploration, and understanding of empirically and/or mentally conceived objects**,

are so closely interwoven, that

any distinction between them is related **more to the point of view adopted** while studying particular aspects of an object, **than to the object itself**

- Arnold (1998):

*The **scheme** of construction of a **mathematical theory** is exactly the **same** as that in any other **natural science***

- Dirac (1970):

*A **theory** with **mathematical beauty** is more likely to be **correct** than an **ugly** one that fits some experimental data*

- Weyl (1922):

*My work always tried **to unite** the **true** with the **beautiful**; but when I had to choose one or the other I usually chose the beautiful*

*Basic **educational** moral/conclusion from Theses **A** to **C***

- By *Thesis* **A**, **history** **cannot be ignored** in **teaching & learning** of either Mathematics, or Physics

- By *Theses* **B & C**, **teaching and learning** **one of them** should take into account, be supported, or include **aspects of the other**

- *Thesis* **C** **gives orientation** to motivate, stimulate, support, deepen, widen **teaching & learning** either discipline, specialized for particular examples into precise guidelines with the aid and/or in the light of Thesis B

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

1. ***Which* history** for **didactical** purposes?

Important (Fried):

Avoid "***Whig***" history (i.e. the present **not be** the **measure** of the past)

History should not be forced "…*to serve aims not only foreign to its own but even antithetical to them*"

In this connection

A useful conceptual pair (Grattan-Guinness):

History – ***He***ritage

complementary perspectives of the historical development

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

***H**istory*: development of knowledge during a **particular** period: its launch, early forms, impact at that period, applications in and/or outside mathematics/physics

- ***What*** *happened in the* ***past***?
- ***Why*** *did it happen?*
- ***What*** *did* ***not happen*** *in the past and* ***why not***?

False starts, missed opportunities…, sleepers, and repeats noted and maybe explained… **differences** from seemingly similar more **modern knowledge**

***He**ritage*: impact of knowledge on **later** work, the forms it may take/be embodied in later contexts. Focus on some of its **modern forms**, with attention to its development… **mathematical relationships noted**, while **historical** ones… hold **less interest**

- ***How*** *did we get* ***here***?

Modern knowledge is **inserted** when appropriate, thereby **unveiling past knowledge**

Similarities with more **modern knowledge** are emphasized;

The present **photocopied** onto the past

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

2. With *which* ***role(s)***?

2.1 **Three** mutually **complementary** **roles**

(Barbin, Furinghetti, Jahnke, van Maanen)

Replacement: Replacing **knowledge** as usually understood (final results; set of techniques; school units for exams etc) by **something different** (deductively organized results, **and** a vivid **intellectual activity** as well).

Reorientation: Changing the "**familiar**", to something "**unfamiliar**"; by **modifying** the conventional **perception of knowledge** as something always existing in its current form, into an **evolving intellectual activity**

Cultural role: Awareness of **knowledge** as an integral part of human **intellectual history**; hence, perceiving **mathematics** and/or **physics** from **perspectives** **beyond** their currently **established boundaries** as disciplines

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

2.2 From the viewpoint of the ***objective*** of integrating ***History*** in Mathematics and/or Physics ***Education***
(ICMI Study)

- ***Learning*** *specific pieces of mathematics and/or physics*
- *Views on the* ***nature of mathematics, physics*** *and the associated activities*
- *The* ***didactical background*** *of teachers and their pedagogical repertoire*
- *The* ***affective predisposition*** *towards mathematics and physics*
- *The* ***appreciation*** *of mathematics & physics as a* ***cultural-human endeavour***

Part 2: The *History* – *Pedagogy* - *Mathematics/Physics* (*HPM/Ph*) **framework** & main **issues**

2.3 From the point of view of the ***way History*** is **accommodated** into ***Education*** (Jankvist)

- ***History*** *as a* ***tool***: History as an **assisting means**, an **aid** in learning & teaching of **mathematics** or **physics**;
  in this sense, history as a **motivational**, **affective**, **cognitive** tool

- ***History*** *as a* ***goal***: History as an **aim in itself**, posing & suggesting answers to questions on the **evolution** and **development** of **mathematics** or **physics**;

- the **inner** & **outer** driving **forces** of this **evolution**;
- **cultural** & **societal aspects** of mathematics or physics and their history

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

3. In *which* ***way(s)*** to be realized in practice?

3. 1 **Three** broad ways to integrate ***History*** in Mathematics & Physics ***Education*** (ICMI Study)

Complementary to each other

Each one emphasizing a different **aim**

- *Provide* ***direct historical information***, **aiming** to **learn history**
- *Implement a* ***teaching*** *approach* ***inspired by history***, **aiming** to **learn mathematics and/or physics**
- *Focus on* ***Mathematics*** *and/or* ***Physics*** *as* ***disciplines*** *and the* ***cultural & social context*** *in which they have been evolving*, **aiming** to **develop deeper awareness** of their
  - evolutionary character,
  - epistemological characteristics,
  - relation to other disciplines,
  - influence by (intrinsic and extrinsic) factors

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

3. In *which* ***way(s)*** to be realized in practice?

3. 2 From a **methodological** point of view (Jankvist)

Illumination approaches: **Teaching** & **Learning** **supplemented** by **historical information** of varying size & emphasis.

Module approaches: **Instructional Units** devoted to **history**, often based on **specific cases**

History-based approaches: **History** **shapes** the order & way of **presentation** - often without appearing explicitly, but rather being - **integrated into teaching**

Part 2: The ***History*** – ***Pedagogy*** - ***Mathematics/Physics*** (***HPM/Ph***) **framework** & main **issues**

Comment:

Approaches may **vary** in **size** & **scope**, according to

- specific **didactical aim**,
- **subject matter**,
- **level** & **orientation** of the learners,
- available **didactical time**,
- **external constraints** (curriculum regulations, number of learners in a classroom etc)

Part 3: **Illustrative** Examples

1. Measuring the distance of inaccessible objects

   *Mathematics & Physics in their **wider cultural context***

2. Rotations, Space-Time & Special Theory of Relativity

   ***How** did we get **here**?*

3. Differential Equations, (Functional) Analysis & Quantum Mechanics

   ***What did** (or did not) **happen** in the past and **why** (or why not)?*

1. Measuring the distance of inaccessible objects

Mathematics & Physics in their ***wider cultural context***

- 1.1 *Eratosthenes' measurement of the Earth's circumference*

- 1.2 *Aristarchus' measurement of the Earth – Sun - Moon relative distances*

- 1.3 *Copernicus' measurement of inner planets' relative distances from the sun*

- 1.4 *Trigonometric parallax* for measuring:

  (i) *Earth-Sun distance* by inner planets' transits across the sun's disk

  (ii) *Star distances* by *stellar parallax*

1. Measuring the distance of inaccessible objects

Mathematics & Physics in their ***wider cultural context***

Rationale

Mathematically, examples (a)-(d) are **elementary**

But

Emphasis: How **elementary geometrical ideas** & **reasoning**

led historically to **astronomically** & **physically** **non-trivial** consequences

with **far-reaching cultural implications** (didactically beneficial)

1. Measuring the distance of inaccessible objects

Placement within the general HPM/Ph framework

Rich example, capable of extension in different directions. E.g.

- A sufficiently self-contained ***interdisciplinary teaching module*** on: elementary Euclidean geometry, modelling of physical situations, astronomical observations, significance of technically accurate instrumentation, crucial role of approximate computations;

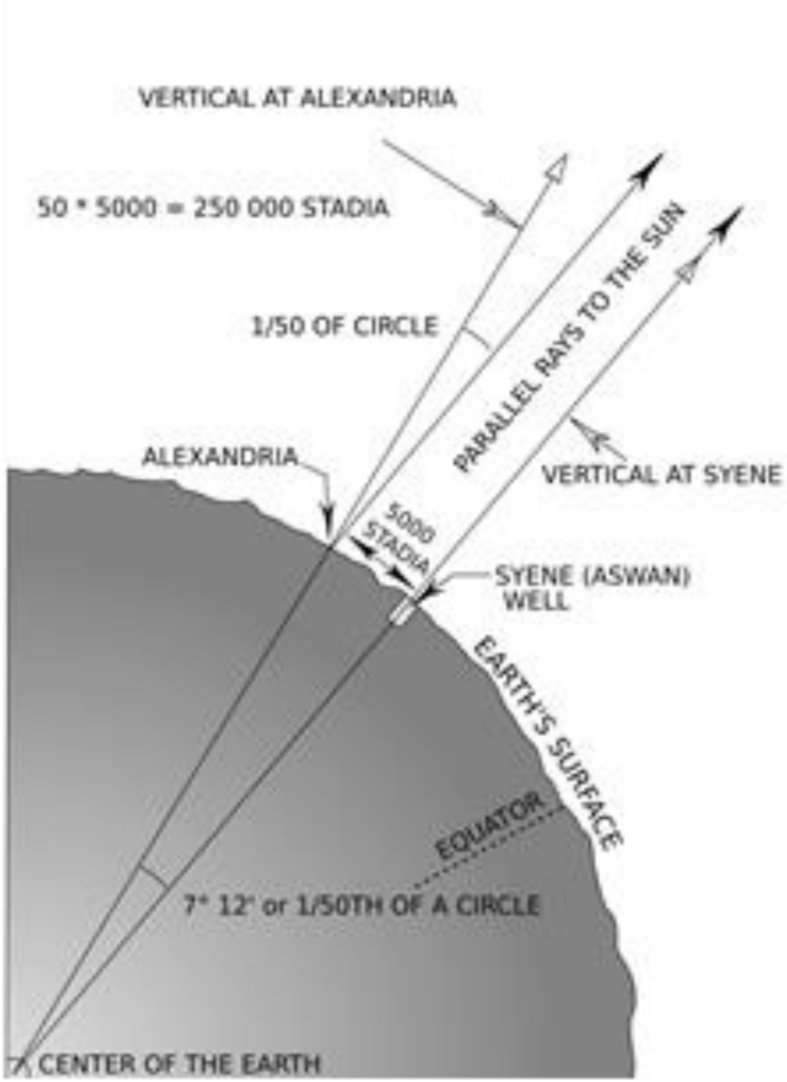
Or

- ***Illuminating*** examples in high-school or university courses on: Euclidean geometry, trigonometry, geography, history of science & math

- A ***heritage-like*** perspective
- ***History*** (mainly) as a ***goal***,
- with a ***cultural role***

\- **bridging mathematics** with other **subjects**,
\- **enriching/widening** teachers' **didactical repertoire**
\- **developing students' awareness**: mathematics & natural sciences in constant dialogue with societal needs & philosophical queries

Eratosthenes' measurement of the Earth's circumference



1. Measuring the distance of inaccessible objects

*Mathematics & Physics in their **wider cultural context***

1.1 *Eratosthenes' measurement of the Earth's circumference*

Three **bold** **non-mathematical** hypotheses:

- (i) ***Earth*** *is* ***spherical***;
- (ii) Alexandria and Syene *lie on the* ***same meridian****;*
- (iii) The ***sun is so far away*** that its light rays are *practically* parallel

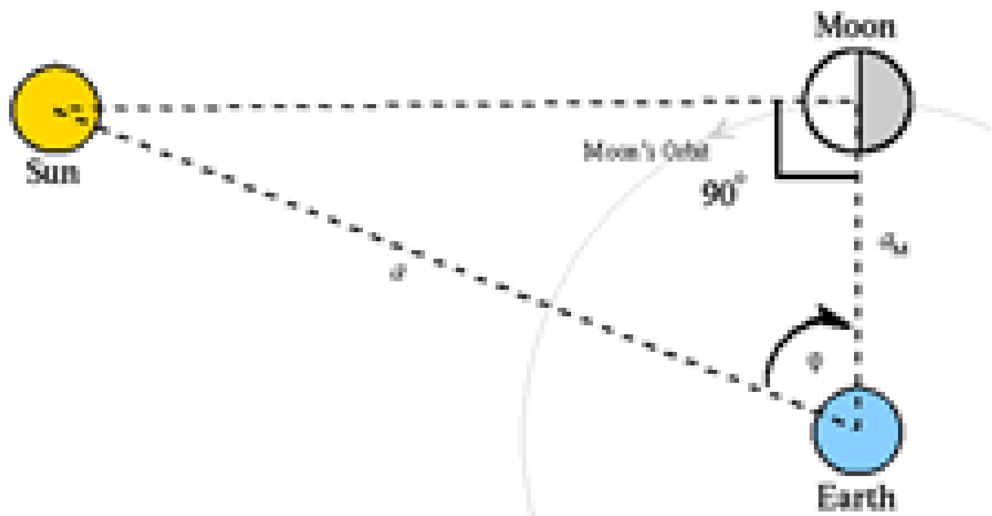
Questions like

- *How do we know that the earth is spherical?*
- *How do we know that two places lie on the same meridian?*
- *How can we check that the sun is really so far away?*

1. Measuring the distance of inaccessible objects

Mathematics & Physics in their ***wider cultural context***

1.2 *Aristarchus' measurement of the Earth – Sun - Moon relative distances*



Trivial mathematics: $\cos\varphi = a_M/a$

But

- Two **bold** **non-mathematical** hypotheses:
- The moon is (i) ***spherical***; (ii) ***illuminated by the sun***

1. Measuring the distance of inaccessible objects

Mathematics & Physics in their ***wider cultural context***

1.2 *Aristarchus' measurement of the Earth – Sun - Moon relative distances*

Remarks:

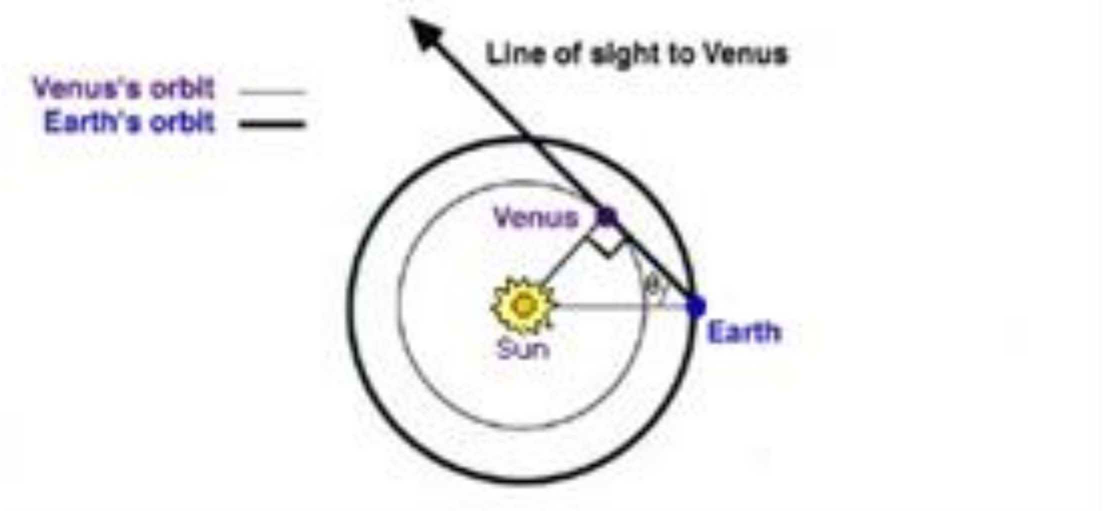
- (1) *How do we know that the moon is* ***spherical****;* ***illuminated by the sun****?*

- (2) Aristarchus' measurement $\varphi^{o} = \mathbf{87^{o}}$, hence $\mathbf{18 < a/a_M < 20}$.
  Actual value: $\varphi^{o} \cong \mathbf{89^{o}52'}$ hence $1/\cos\varphi \cong a/a_M \cong \mathbf{390}$
  - limited ***accuracy*** of ***instruments***
  - *sensitive* ***dependence*** *of* ***computations*** *on* ***data***

1. Measuring the distance of inaccessible objects

Mathematics & Physics in their ***wider cultural context***

1.3 *Copernicus' measurement of inner planets' relative distances from the sun*



Trivial mathematics:

At **greatest** angular **elongation** from the sun: $\boldsymbol{a_P = a \sin\theta}$

But

- Two **bold non-mathematical** hypotheses:

inner ***planets revolve*** *around the* ***sun*** in ***circular*** orbits

1. Measuring the distance of inaccessible objects

Mathematics & Physics in their ***wider cultural context***

1.3 *Copernicus' measurement of inner planets' relative distances from the sun*

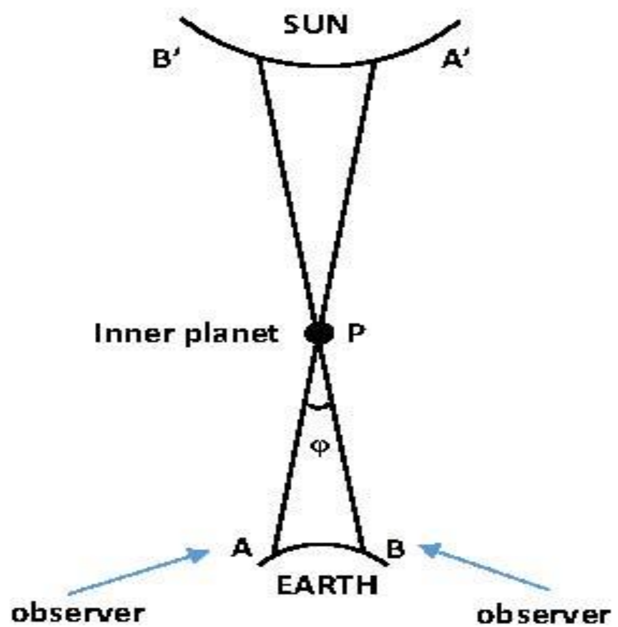
Remarks:

- (1) What was ***Copernicus' motivation*** *of such a* ***bold****,* ***counter-intuitive*** *assumption*?
- (2) ***Existence*** *of* ***greatest*** *elongation of the* ***inner*** *planets*, fitted ***naturally*** in Copernicus' system, but **not** in Ptolemy's
- (3) Discuss ***Tycho Brahe's semi-heliocentric system*** & ***Kepler's heliocentric system of elliptic orbits***

1. Measuring the distance of inaccessible objects

1.4 *Trigonometric parallax*

(i) *Earth-Sun distance* by inner planets' transits across sun's disk



A **bold** **non-mathematical** hypothesis: *planet & sun very far away*

Hence, by **trivial mathematics**: **sin*φ* ≅ *φ* ≅ *AB/PA*** $(\varphi^{o}<0^{o}.5)$

PA/A'A ≅ ***PB/B'B*** known by ***Copernicus'*** method, or ***Kepler's 3rd law*** *of planetary motions*

1. Measuring the distance of inaccessible objects

1.4 *Trigonometric parallax*

(i) *Earth-Sun distance* by inner planets' transits across sun's disk

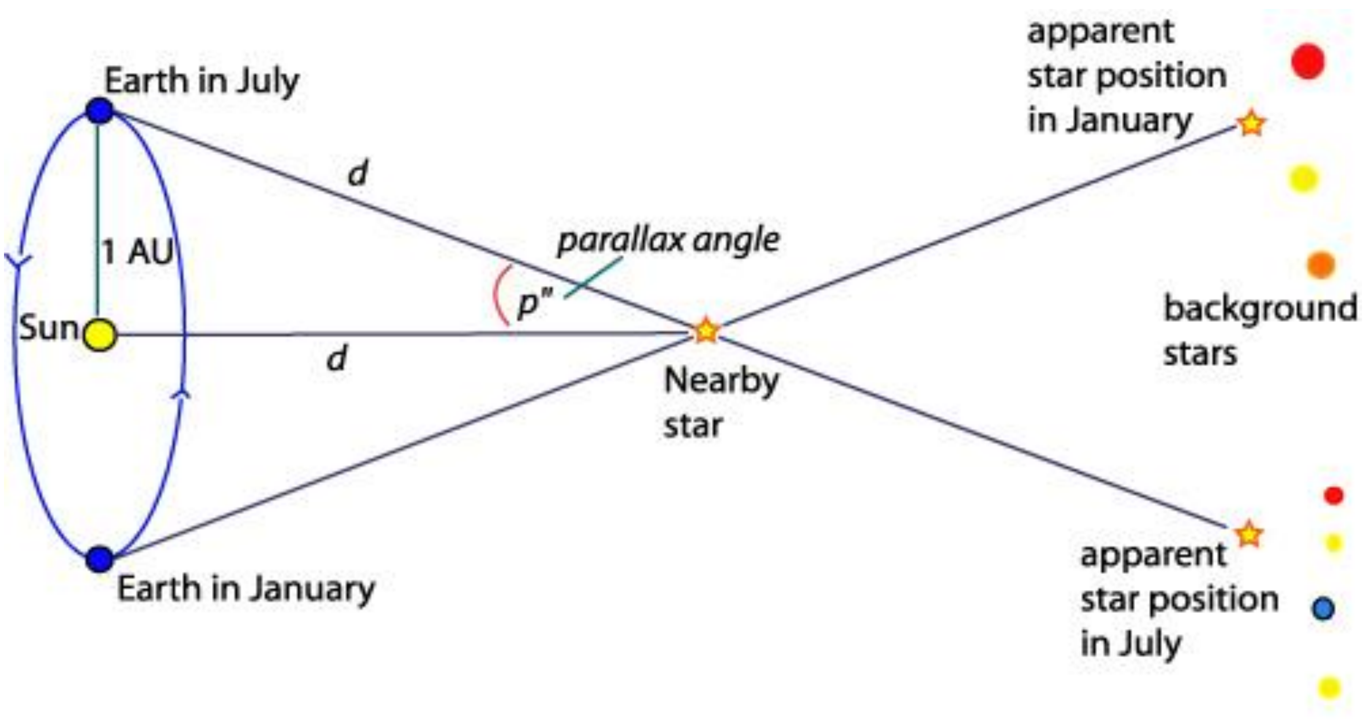
Remarks:

- ***Earth-sun distance*** gives ***meaning*** to all celestial ***relative* distances** got by *other methods*
- ***Venus***' *transits* are ***rare*** (110 years). ***Mercury****'s* frequent (every few years) but ***less favorable***
- Today we use Modern ***radar methods*** directly (***conceptually* simple** - ***technically* sophisticated**)

1. Measuring the distance of inaccessible objects

1.4 *Trigonometric parallax*

(i) *Star distances* by *stellar parallax*



Trivial mathematics: $p < 1''$ **hence** $a \cong d \sin p$

so $d = (206{,}265/p'')$AU (AU ≡ 1.49×10^{8}km)

1. Measuring the distance of inaccessible objects

1.4 *Trigonometric parallax*

(i) *Star distances* by *stellar parallax*

But

Two **bold non-mathematical** hypotheses:

- (i) *Earth revolves around the sun*;
- (ii) *Faint stars* (statistically) *far away*: A sufficiently *immovable background*

Remarks:

- ***Conceptually*** **simple**, but ***technically*** **sophisticated** idea of ***parallax*** was used by Copernicus' & Galileo's *Aristotelian opponents* ***against earth's motion***; **no** parallax observed
- ***Technically possible*** *measurement of parallax* after the telescope as **late** as **1838** (Bessel); a *definite experimental test of earth's revolution*
- ***Other*** *such* ***tests****?*

2. Rotations, Space-Time & Special Theory of Relativity

How *did we get* ***here?***

Rationale

- ***Special Relativity*** **standard** for **Physics** undergraduates
- ***Matrix/linear algebra*** **standard** for **Mathematics/Physics** undergraduates

- **Power of algebra**: **unification-through-abstraction** of distinct **concrete** problems,

- Undergraduates students meet grave **difficulties** in studying **abstract algebraic concepts**, because of limited mathematical maturity

- **Hence**: **algebraic concepts** should be taught using **concrete meaningful** examples

2. Rotations, Space-Time & Special Theory of Relativity

How *did we get* ***here****?*

Placement within the general HPM/Ph framework

Fairly complete account of **Special Relativity**'s **foundations**

Minkowski's original **ideas** on **space-time**, using simple **matrix algebra**

- (Mainly) a *heritage-oriented*, *illumination approach*
- *inspired by & based on* ***history*** with
- **history**
  - having a ***re-orientation*** *role*
  - serving mainly *as a* ***tool***
- for ***learning mathematics & physics*** by
  - *unfolding their interplay*
  - *enriching teachers' didactical repertoire*

2. Rotations, Space-Time & Special Theory of Relativity

How** did we get **here?

Link **innovations** in **Physics** & their **mathematical formulation**, to their **modern counterparts**:
thus illuminating ***how did we get here?***

Key historical elements

physically-oriented approach

Lorentz (1904)
Einstein (1905)

transformations leaving invariant *Maxwell's equations*

- analysis of ***simultaneity***
- *Special **Relativity Principle***
- Constancy *of **light speed***

mathematically-oriented approach

Poincaré (1904)

group structure of transformations between *inertial systems*

Minkowski (1908)

- the ***space-time*** concept
- transformations leaving invariant the *light cone*:

$$x^2 + y^2 + z^2 - c^2 t^2 = 0$$

2. Rotations, Space-Time & Special Theory of Relativity

How *did we get* ***here?***

Sketch of a possible didactical implementation

- **LT** in (x, t)-plane in close analogy with **plane rotations** in (x, y)-plane:

Rotations R_φ conserve Euclidean distance x^2+y^2

$$R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

LT L_φ conserve Minkowski pseudo-distance x^2-$c^2\, t^2$ (tanh$\varphi = v/c$ $|v/c|<1$)

$$L_\varphi = \begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}$$

- **Group structure**:

Successive **rotations** by angles φ, φ' ≡ rotation by $\varphi+\varphi'$: $\mathbf{R_\varphi R_{\varphi'} = R_{\varphi+\varphi'}}$

Successive **LT** by angles φ , φ' ≡ LT by $\varphi+\varphi'$: $\mathbf{L_\varphi L_{\varphi'} = L_{\varphi+\varphi'}}$

2. Rotations, Space-Time & Special Theory of Relativity

How *did we get* ***here****?*

Sketch of a possible didactical implementation

Hence

geometry/matrix algebra: $$\tanh(\phi+\phi') = \frac{\tanh\phi + \tanh\phi'}{1+\tanh\phi\tanh\phi'}$$

⇔ physics: $$v'' = \frac{v+v'}{1+\frac{1}{c^2}vv'}$$ (relativistic velocity addition)

⇔ algebra: $$x \bullet x' = \frac{x+x'}{1+xx'}$$ ((-1,1], •) commutative group

$x \bullet 1 = 1$, $x = v/c$

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

What *did (or did* ***not****)* ***happen*** *in the* ***past*** *&* ***why*** *(or* ***why not****)?*

Rationale

Subjects **taught** to (under)graduates, **separately** as **heterogeneous**, often **unmotivated**

- ***Jacobi's method*** to solve 1st order PDE
- ***Hamiltonian formulation*** *of Classical Mechanics* (CM) & *Hamilton-Jacobi* theory
- **Analogy**: Fermat's ***Least Time Principle*** (GO) & Maupertuis' ***Least Action Principle*** (CM)
- ***Schrödinger's equation*** in Quantum Mechanics (QM)
- Heisenberg's ***Matrix Mechanics*** & infinite dimensional matrices
- Infinite dimensional linear spaces; (separable) ***Hilbert spaces***
- ***Observables*** in QM as ***self adjoint*** operators; their non-commutative algebraic structure
- ***Fourier*** analysis, ***Lebesgue*** integration, ***squarely-integrable functions***

Historically **strong interconnections** that motivated, stimulated, guided the **development** to their current form; **beneficial** for teaching & learning

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

What *did (or did* ***not****)* ***happen*** *in the* ***past*** *&* ***why*** *(or* ***why not****)****?***

Placement within the general HPM/Ph framework

Vast, rich subject

- *A* ***history-based approach***, ***inspired by history***
- History having a ***replacement*** *role*, serving mainly ***as a tool*** for

- ***learning*** *mathematics & physics*
- ***enriching*** *teachers'* ***didactical repertoire***
- amending students' ***affective predisposition*** towards abstract/difficult concepts

Emphasis on a ***history-oriented*** approach, to enlighten "***what and why did/did not happen?***"

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

What *did (or did* ***not****)* ***happen*** *in the* ***past*** *&* ***why*** *(or* ***why not****)?*

Key historical elements

- 18th century: Maupertuis' *Least Action Principle* (CM) **motivated by/in analogy** with Fermat's *Least Time Principle* (GO)

- 1830s: Hamilton's **mathematically unified** treatment of CM & GO

  Hamilton-Jacobi method to solve 1st order PDEs ⇔ powerful ***new formulation of CM***

- Mathematical **"isomorphism" CM ≈ GO**

  1924: de Broglie → ***wave-particle duality*** (microcosm)

  1926: Schrödinger → ***Wave Mechanics***

- 1925, Heisenberg: **Fourier series** operations **&** atomic **spectroscopy**'s empirical data → ***Matrix Mechanics***, *Heisenberg's* ***indeterminacy relations***

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

What *did (or did* ***not****)* ***happen*** *in the* ***past*** *&* ***why*** *(or* ***why not****)****?***

Key historical elements

- **Both** *Wave Mechanics* & *Matrix Mechanics* **compatible with experiment**; but **conceptually/mathematically** completely **different**!

- 1926, Schrödinger's **formal** proof of their **equivalence**:

Functions in ***Wave Mechanics*** elements of $L^2(\boldsymbol{R})$ with scalar product

$$< f, g >= \int_{-\infty}^{\infty} f^* g d\mu$$

Matrices in ***Matrix Mechanics*** acting on the infinite-dimensional linear space $\boldsymbol{l^2}$ (complex sequences $\alpha=(\alpha_1, \alpha_2, \ldots)$, with $\sum_k |\alpha_k|^2<+\infty$)

- In an orthonormal basis of $L^2(\boldsymbol{R})$,

solving Schrödinger's PDE (*Wave Mechanics*) $\Leftrightarrow$
solving eigenvalue problem for Hamiltonian matrix (*Matrix Mechanics*)

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

***What** did (or did **not**) **happen** in the **past** & **why** (or **why not**)?*

Key historical elements

- 1927ff, von Neumann: **rigorous** proof of their **equivalence**:

- Identified the **algebraic properties** of the objects of the **two theories**,

- Emphasized the **linear structure** of the **function spaces** ($L^2(\boldsymbol{R})$ & l^2) underlying them

- Introduced **axiomatically** the concept of ***separable Hilbert space*** (two examples being $L^2(\boldsymbol{R})$ & l^2)

- Proved **isomorphism** of **all** ***separable Hilbert spaces***

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

What *did (or did* ***not****)* ***happen*** *in the* ***past*** *&* ***why*** *(or* ***why not****)****?***

Sketch of a possible didactical implementation

I. *Least Action Principle* & *Least Time Principle*:

- **Important** examples of **variational principles**, leading to key results in
- Classical Mechanics (*Hamilton-Jacobi equation*), Geometrical Optics (*eikonal equation*)
- **Generic** examples to establish: solution of 1st order PDEs ***equivalent*** to solution of a system of 1st order ODEs the associated *canonical* (*Hamilton*'s) equations

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

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Sketch of a possible didactical implementation

II. From *Classical Mechanics* to *Wave Mechanics:* The *optical analogy*

Fermat's Principle $\delta l = \delta \int_A^B n\,ds = 0$

Least Action Principle $\delta S = \delta \int_A^B \sqrt{2(H-V)}\,ds = 0$

Geometrical Optics

Eikonal equation $|\nabla l|^2 = n^2$

By analogy

Classical Mechanics

Hamilton-Jacobi equation $|\nabla S|^2 = 2(H-V)$

Hamilton's unified treatment (1833-35)

Schrödinger's basic idea (1926)

approximation

approximation

Wave Optics

By analogy

(*New*) **Wave Mechanics**

wave equation for light

new wave equation $i\sigma \frac{\partial \psi}{\partial t} = H(x, i\sigma \frac{\partial}{\partial x})\psi$

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

***What** did (or did **not**) **happen** in the **past** & **why** (or **why not**)?*

Sketch of a possible didactical implementation

- **Why** Hamilton did **not** formulate ***Wave Mechanics***?
- **Extra condition** needed/missing to **give meaning to** $\boldsymbol{\sigma}$

Crucial idea:

de Broglie's postulated the **wave nature** of **matter** by "symmetrising" the Planck-Einstein conception of the **corpuscular nature** of **radiation**

$$E = h\nu,\ \mathbf{p} = h\mathbf{k},\ (h \text{ Planck's constant}) \Longrightarrow \sigma \equiv h$$

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

***What** did (or did **not**) **happen** in the **past** & **why** (or **why not**)?*

Sketch of a possible didactical implementation

III. Heisenberg formulation of *matrix mechanics*: Reasoning by *analogy*

Classical positions & momenta q, p

Fourier representation
Singly indexed frequencies
$\nu_k = k\omega$

$\nu_k + \nu_l = \nu_{k+l}$

$q(t) = \sum q_k \exp(i\, \nu_k t)$

Quantum positions & momenta q, p

(looking for a) ***new*** representation (because of)
Doubly indexed spectral frequencies
ν_{nm}
obeying ***Ritz principle***
$\nu_{nl} + \nu_{lm} = \nu_{nm}$

$q(t) \sim q_{nm} \exp(i\, \nu_{nm} t)$

by analogy ⟷

Operations
by analogy

$p+q \rightarrow (p_k+q_k) \exp(i\, \nu_k t)$ ⟷ $p+q \rightarrow (p_{nm}+q_{nm}) \exp(i\, \nu_{nm} t)$

$pq \rightarrow (\sum_l p_l\, q_{k-l}) \exp(i\, \nu_k t)$ ⟷ $pq \rightarrow (\sum_l p_{nl}\, q_{lm}) \exp(i\, \nu_{nm} t)$

by analogy + Ritz Principle

$$qp - pq = \frac{h}{2\pi i} I$$

(hence, $pq \neq qp$ leading to Heisenberg's **uncertainty relations**)

3. Differential Equations, (Functional) Analysis, Quantum Mechanics

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Sketch of a possible didactical implementation

IV. Introducing concepts & results of ***Functional Analysis***

For instance:

(i) Present the **mathematical problem** of ***Matrix Mechanics*** & ***Wave Mechanics***:

- to diagonalize the Hamiltonian matrix in l^2,
- to solve Schrödinger's PDE equation in $L^2(\boldsymbol{R})$

(ii) Naturally ask for the **relation** of the physically & mathematically **a priori different** theories that yield **identical**, **experimentally correct predictions**

- Give Schrödinger's **heuristic**/non-rigorous **formal proof** of their **equivalence** ($l^2 \approx L^2(\boldsymbol{R})$)
- Prove **rigorously** that l^2 and $L^2(\boldsymbol{R})$ are **isometric Hilbert spaces**

(iii) Reverse the argument and **prove** the **isomorphism** of linear spaces with a scalar product spanned by a countable ON basis (von Neumann's approach)

(iv) **Introduce** other important **concepts**/results: *bounded* vs *unbounded operators; hermitian* as distinct from *self-adjoint operators; extension of an operator* etc

Final Comments

In this talk I tried

- to look at the **innermost relationship** of **Mathematics** & **Physics** considered both from the point of view of their **epistemological characteristics**; **historical development**, summarized in **three** main ***theses*** implying (from an educational point of view) that **this relationship** should be **taken into account** explicitly
- to **address** the **main issues** faced in any **such attempt** and
- to describe a **common framework** for integrating **history** into **teaching & learning both** disciplines
- to **illustrate** these general ideas by three **examples** of different content and orientation

Hopefully, enough evidence has been presented to support that

(a) It is **impossible** to deeply **understand** either Mathematics or Physics without being sufficiently aware of their **interconnections** and mutual **influence**;

(b) On the contrary, taking into account their rich **interrelation** is **beneficial** for **teaching** & **learning** either discipline

THANK YOU

for your attention & patience